

Energy Dissipation in Hamiltonian Chains of Rotators

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Abstract. We discuss, in the context of energy flow in high-dimensional systems and Kolmogorov-Arnol'd-Moser (KAM) theory, the behavior of a chain of rotators (rotors) which is purely Hamiltonian, apart from dissipation at just one end. We derive bounds on the dissipation rate which become arbitrarily small in certain physical regimes, and we present numerical evidence that these bounds are sharp. We relate this to the decoupling of non-resonant terms as is known in KAM problems.

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1. Introduction

In this paper, we consider a chain of coupled rotators. Our basic system is Hamiltonian, and we are specifically interested in the way in which energy is transported through the system. We study this by adding dissipation to the first rotator, and then considering initial conditions in which the energy of the system is localized in a rotator far from the one which dissipates energy. Our main result is a lower bound on the energy dissipation rate. The lower bound becomes arbitrarily small as either the energy of the fast rotator is made very large, or as the number of rotators increases. Although at the moment our rigorous estimates are “one-sided”, we present numerical evidence that they are sharp, in the sense that in some regimes energy dissipation does scale like our lower bound, indicating that energy transport in the system is indeed extremely slow.

Hamiltonian systems, non-equilibrium problems and the interplay of rigorous results with computer based insight have been central themes in “Nonlinearity” since its origin 30 years ago and as such we feel this paper is an appropriate way to celebrate this anniversary. We thus use this introduction to sketch a (relatively personal) view on some highlights in the subject of our paper, with emphasis on papers which have appeared in *Nonlinearity*.

We start with Hamiltonian systems: Here, our problem is connected with ideas like the KAM and Nekhoroshev theorem which have appeared repeatedly in *Nonlinearity*. [5, 31, 29, 28, 27, 33, 30]. The common feature of both of these approaches is the realization that in certain limiting regimes (usually small coupling) Hamiltonian systems are close to their decoupled variants. One typically takes advantage of this by transforming the system to some sort of normal form by canonical transformations. In the particular problem considered in this paper the situation is more reminiscent of the Nekhoroshev than KAM theorems since we will also make a finite number of transformations of our original system to bring it to a form in which the dissipation can be directly estimated. Another key idea is periodic orbits play an especially important role in the stability of systems with respect to perturbations. Early papers in this context are [31, 29, 33]. In particular, Lochak pointed out that the evolution of Hamiltonian systems in the neighborhood of periodic orbits occurred only on a very long time scale. Of particular interest in our context, are the spatially localized periodic orbits known as “breathers” [30]. One respect in which our problem differs from this preceding work is that we do not deal with small coupling, but with high frequencies. It seems this problem is not

readily transformed into one with small coupling, but is rather close to ideas having to do with breathers, isolated high-energy states in extended systems [34, 35, 20, 21, 11, 10, 12].

We find that our system approaches a “quasi-stationary” state in which the rotors oscillate with an amplitude that decreases in a regular way from the rotator in which the original energy was concentrated, to the dissipating rotor. These states are reminiscent of breather solutions in Hamiltonian lattices [30, 1, 3, 2] which are known to be stable over long time intervals even in the purely Hamiltonian setting. That these breather states should inhibit transport in coupled lattice systems has already been suggested by several authors [34, 35, 20, 21].

They are also reminiscent of metastable states in weakly viscous fluid flows where invariant structures present in the conservative limit (*i.e.*, Euler equations) are replaced by families of solutions which seem to determine the long relaxation toward the rest state when weak dissipation is introduced [4, 19]. In a similar fashion, in our problem, the breather solutions which are exact, periodic solutions in the conservative limit seem to be replaced by nearly periodic dissipative solutions along which the solution evolves toward the rest state.

The combination of such problems with dissipation is more recent: A careful study was done by [35, 13]. These studies have in common that they estimate the rate of dissipation. Our goal in the present work is more restricted than these prior studies. We are particularly interested in identifying the states of the system which give rise to the slowest possible dissipation rate. Because we consider problems in which the initial energy is localized far from the center of dissipation, these correspond to states in which the transport of energy through the lattice is especially slow. In another direction dissipation in Hamiltonian systems has become an intensive object of study in planetary studies [8, 9, 26].

In [9, 26] the authors study the evolution of trajectories in nearly integrable Hamiltonian systems subjected to very general, small dissipative forces. They show that solutions remain close to the integrable solutions for long times by using a combination of canonical transformations, and then representing the dissipation in the canonically transformed variables.

Another, related, direction are studies of out-of-equilibrium systems, usually in a stochastic context (the stochasticity modeling heat-baths) [16, 15, 34, 35, 18, 17, 14]. In most of these papers, one of the problems is to show that transport actually happens, and that the energy of the system does not blow up due to the energy input from the heat baths [15, 20]. There is now a quite abundant literature about non-equilibrium steady states of simple, 1-dimensional systems, where masses or rotators are coupled to heat baths [22, 7, 6]. One of the issues in this context is to prove the existence of stationary states for such systems. It has been recognized early on that the convergence for “arbitrary” initial conditions to such stationary states can be very slow, in particular, when one of the masses or rotors has very high energy [24, 25]. This phenomenon is rightly attributed to a loss of effective coupling between a high energy site and its low energy neighbors, because of an averaging effect of the rapidly oscillating force. Combining some of the two aspects above [13] studied heat transport in a chain of oscillators, in the limit of weak coupling (see also [23]).

In this paper, we address a much simpler problem, namely a purely deterministic system (*i.e.*, without any stochastic heat baths) but only with dissipation and an otherwise purely Hamiltonian interaction. Obviously, such a system will eventually converge to the state where

nothing moves, but we are interested in the initial phase of such a system where the energy is still very high and the system loses energy (very slowly) through the dissipation. The advantage of considering such a simple system is the possibility of understanding it quite completely.

More precisely, we study in detail the behavior of a chain of n coupled rotators (with *nearest neighbor coupling*) with *localized dissipation on the first of these rotators*. We are interested in the evolution of this system in a region of phase space where almost all of the energy of the system is located in one of the sites. Such a setup is of course reminiscent of breathers, but now we have dissipation.

In particular, our results are consistent with the suggestion put forth by [24] that even the notion of stationary state in large (infinite) such systems might need a new definition. Because certain regions of phase space (*e.g.*, the states we study in which the initial energy is localized far from any source of dissipation) take so long to converge to the stationary state, the actual limiting state may not be the most relevant quantity for practical effects like heat conduction or the Fourier law.

2. Model and statement of results

We fix the number $n > 1$ of rotators once and for all, and use the phase space $\Omega \equiv \mathbf{R}^n \times \mathbf{T}^n$ for the actions $\mathbf{I} = (I_1, \dots, I_n)$ and the angles $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$. We adopt the convention $\mathbf{T} = \mathbf{R}/2\pi$. We shall often write $x = (\mathbf{I}, \boldsymbol{\varphi})$. We consider Hamiltonians of the form

$$H^{(0)}(\mathbf{I}, \boldsymbol{\varphi}) = \frac{1}{2} \sum_{i=1}^n I_i^2 + \sum_{i=1}^{n-1} U_i(\varphi_{i+1} - \varphi_i) = h^{(0)}(\mathbf{I}) + f^{(0)}(\boldsymbol{\varphi}) , \quad (2.1)$$

where the functions $U_i : \mathbf{T} \rightarrow \mathbf{R}$ are real, smooth interaction potentials subject to the following two assumptions.

Assumption 2.1. *The Fourier series of each U_i contains finitely many modes.*

Assumption 2.2. *Each U_i is non-degenerate in the sense that there is no $\varphi \in \mathbf{T}$ for which $U'_i(\varphi) = U''_i(\varphi) = 0$. In other words, the interaction force and its derivative never vanish simultaneously.*

Without loss of generality, we also assume that the U_i contain no constant (and physically irrelevant) Fourier mode.

We fix a dissipation coefficient $\gamma > 0$ and consider, for $i = 1, \dots, n$, the evolution equations

$$\begin{aligned} \dot{I}_i &= -\partial_{\varphi_i} H^{(0)} - \delta_{i,1} \gamma I_1 , \\ \dot{\varphi}_i &= \partial_{I_i} H^{(0)} . \end{aligned} \quad (2.2)$$

We fix $2 \leq k \leq n$. We will concentrate on domains in phase space where I_k is of order $L \gg 1$, and the other actions are of order $\varrho \sim 1$. Without loss of generality, we will consider only the case $L > 0$, since the case $L < 0$ is identical by symmetry.

Definition 2.3. For $L, \varrho > 0$, we denote by $\mathcal{B}_{L,\varrho}$ the open set

$$\mathcal{B}_{L,\varrho} = \{(\mathbf{I}, \boldsymbol{\varphi}) \in \Omega : |I_i - L\delta_{i,k}| < \varrho, \ i = 1, 2, \dots, n\} .$$

Our main result is

Theorem 2.4. Consider the system of equations (2.2), under Assumptions 2.1 and 2.2. Fix $\alpha > 0$ and $\varrho > 0$. Then, there exist constants $L_0, \varrho^*, C_1 > 0$ such that for all $L \geq L_0$ and all initial conditions $x(0) \in \mathcal{B}_{L,\varrho}$, the following holds for $T \equiv \alpha L^{2k-3}$:

(i) We have

$$x(t) \in \mathcal{B}_{L,\varrho^*}, \quad 0 \leq t \leq T .$$

(ii) Moreover,

$$H(x(T)) - H(x(0)) \leq -C_1 \frac{T}{L^{4k-6}} . \quad (2.3)$$

Remark 2.5. Part (ii) says that the amount of energy loss per unit of time is at least proportional to L^{6-4k} , when averaged over times of order L^{2k-3} . Numerical investigation (see Sect. 3) indicates that the power $4k - 6$ in the theorem is optimal.

Remark 2.6. Assumption 2.2 is crucial for the theorem above: as we will show in Sect. 9, dropping Assumption 2.2 may lead to decay rates that are much slower than (2.3).

2.1. The relation to KAM

Part of our study is closely related to the usual techniques of KAM theory. However, the addition of dissipation to an otherwise purely Hamiltonian problem requires significant modifications of the classical KAM machinery.

In general, KAM-like calculations are done for weak coupling. In our case, the coupling is of order 1 but the energies are very high. It is tempting to envisage a scaling which connects our problem to the well-studied KAM techniques. We have not been able to find such a scaling, probably due to the fact that rotators are akin to pinning potentials of infinite power (i.e., potentials like x^{2m} with $m = \infty$).

3. Phenomenology

In the regime of interest, the k 'th rotor holds most of the energy of the system, and tends to effectively decouple from its neighbors (or its neighbor if $k = n$), as the interaction forces involving the k 'th site oscillate very rapidly and average out.

The idea of the proof of Theorem 2.4 is as follows. In Sect. 5 and Sect. 6, we will construct new canonical coordinates $\tilde{x} = (\tilde{\mathbf{I}}, \tilde{\boldsymbol{\varphi}})$, which are very close to x when L is large, and which make the decoupling explicit. More precisely, the coordinates \tilde{x} obey some dynamics where the site k is decoupled from the rest of the system, except for some “remainders” of magnitude $L^{2-2k} \ll 1$ (see (6.10)). Part (i) will follow from this and technical considerations (see Proposition 6.9).

Moreover, we will obtain (see Remark 6.2) the scaling

$$\begin{aligned}\tilde{I}_i - I_i &\sim L^{1-2|k-i|} & (i \neq k), \\ \tilde{I}_k - I_k &\sim L^{-1}, \\ \tilde{\varphi}_i - \varphi_i &\sim L^{-2|k-i|} & (i \neq k), \\ \tilde{\varphi}_k - \varphi_k &\sim L^{-2}.\end{aligned}\tag{3.1}$$

In particular, we have

$$I_1 = \tilde{I}_1 + P_1,$$

where P_1 scales like L^{3-2k} . To prove Part (ii) of Theorem 2.4, we will use the decomposition

$$\frac{d}{dt}H = -\gamma I_1^2 = -\gamma \tilde{I}_1^2 - \gamma P_1^2 - 2\gamma \tilde{I}_1 P_1.\tag{3.2}$$

The key role will be played by the second term in the right-hand side (see Sect. 7.1), which is of order L^{6-4k} and will give rise to the dissipation rate in (2.3). The third term has no sign, and we will show in Sect. 7.2 that it has negligible effect thanks to its oscillatory nature. It is the comparison of the second and third contributions in (3.2) that leads to the choice $T \sim L^{2k-3}$ (see Remark 7.2).

In addition to being the central ingredient of our proof of Theorem 2.4, the construction of the new coordinates \tilde{x} leads to some very interesting observations, which we illustrate here without proof.

First, we observe numerically (see Sect. 3.1) that the system quickly reaches a quasi-stationary state, where the actions I_i oscillate with frequency of order L and small amplitude around some fixed value $\langle I_k \rangle \approx L$ for I_k , and around zero for I_i , $i \neq k$. At the same time, φ_k rotates rapidly, while the angles φ_i , $i \neq k$ oscillate with small amplitude around an equilibrium position of the interaction potentials. More precisely, the amplitude of the oscillations of the actions appears to scale like

$$\begin{aligned}I_i &\sim L^{1-2|k-i|} & (i \neq k), \\ I_k - \langle I_k \rangle &\sim L^{-1}.\end{aligned}\tag{3.3}$$

This can be understood as follows. By construction, the action \tilde{I}_k is approximately constant, which with (3.1) explains (3.3) for site k . To explain (3.3) for $i < k$, we observe that the system $(\tilde{\varphi}_1, \dots, \tilde{\varphi}_{k-1}, \tilde{I}_1, \dots, \tilde{I}_{k-1})$ has, up to small corrections, the dynamics of a chain of $k-1$ rotors with dissipation on the first one, and initial energy of order 1. As there is no decoupling phenomenon among these $k-1$ “slow” rotors, this subsystem rapidly dissipates most of its energy and approaches some rest position where $\tilde{I}_i \approx 0$ for all $i < k$, and where all the $\tilde{\varphi}_i$, $i < k$, are close to a local equilibrium of the interaction potentials. This and (3.1) explain (3.3) for $i < k$.

Explaining why the same holds for $i > k$ (in case $k < n$) is less obvious. In fact, while the variables $(\tilde{I}_k, \tilde{\varphi}_k)$ are decoupled from the rest of the system up to terms of order L^{2-2k} , the two subsystems $(\tilde{\varphi}_1, \dots, \tilde{\varphi}_{k-1}, \tilde{I}_1, \dots, \tilde{I}_{k-1})$ and $(\tilde{\varphi}_{k+1}, \dots, \tilde{\varphi}_n, \tilde{I}_{k+1}, \dots, \tilde{I}_n)$ still interact

through some terms of order L^{-2} (see Remark 6.1), which is much larger than L^{2-2k} as soon as $k \geq 3$. This seems to allow the energy of the subsystem $(\tilde{\varphi}_{k+1}, \dots, \tilde{\varphi}_n, \tilde{I}_{k+1}, \dots, \tilde{I}_n)$ to dissipate rather rapidly (*i.e.*, much faster than any significant change in the energy of rotator k can be observed) so that the argument above also yields (3.3) for $i > k$.

Coming back to (2.3), the following can be said. During the initial transient phase, the instantaneous dissipation rate can be much larger than L^{6-4k} . However, once the quasi-stationary state is reached, then by (3.3) we really have $\frac{d}{dt}H = -\gamma I_1^2 \sim -L^{6-4k}$, indicating that the power of L in (2.3) is optimal (in the sense that choosing initial conditions that are already in the quasi-stationary state yields a dissipation rate which is no faster than what we claim).

3.1. Numerical illustration

We illustrate here the properties of the quasi-stationary state for short chains, assuming that $U_i(\varphi_{i+1} - \varphi_i) = -\cos(\varphi_{i+1} - \varphi_i)$ for all i . We first consider $n = k = 4$ and $\gamma = 1$, with $L = 10$ and $L = 100$. For all numerical simulations, we choose the initial condition $x_0 \in \mathcal{B}_{L,q}$ such that $I_i(0) = 0$ for $i \neq k$, $I_k(0) = L$, and $\varphi_i(0) = 0$ for all i . In Fig. 1, we depict the maximum of $|I_i|$ over intervals of length $2\pi/L$ (corresponding to periods of φ_4).

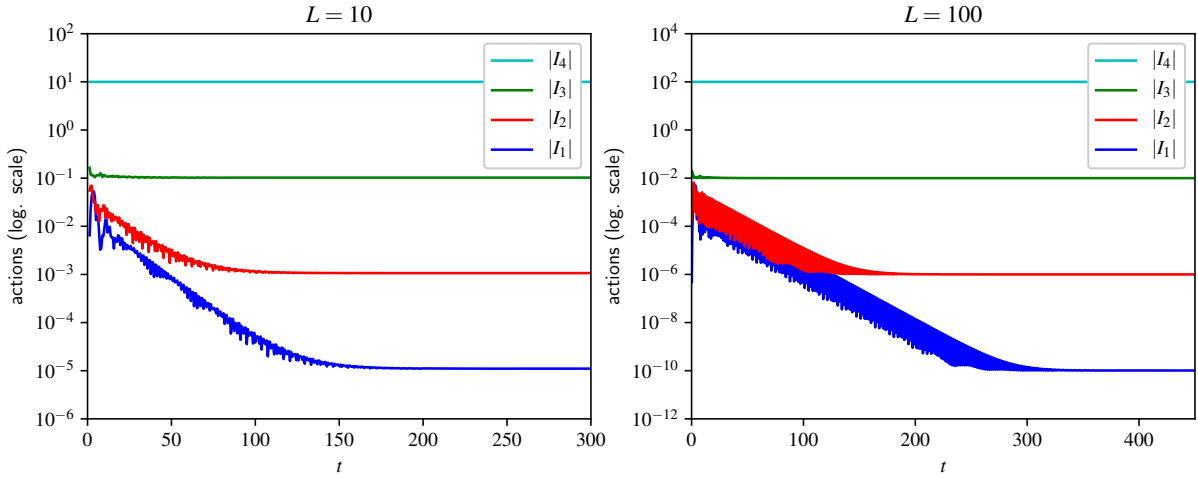


Figure 1. Maximum of $|I_i|$ over intervals of length $2\pi/L$ for $n = k = 4$ and $\gamma = 1$, with $L = 10$ (left) and $L = 100$ (right).

The transient phase and the quasi-stationary state are clearly visible on Fig. 1. Observe in particular that the I_i “equilibrate” one after the other, with I_1 needing the longest time. Note that the vertical axis is $\log |I_i|$ and that the observed values in the quasi-stationary state scale like $|I_i| \sim L^{1-2(n-i)}$, which is (3.3). The equidistance of the lines reflects the relation $\frac{|I_i|}{|I_{i+1}|} \sim \frac{1}{L^2}$. The agreement here is perfect because for cosine potentials the numerical prefactors in (3.3) happen to be 1 (see Sect. 8), so that the I_i , $i \neq k$, are expected to oscillate with amplitude $(1 + \mathcal{O}(L^{-1}))L^{1-2(n-i)}$.

Fig. 2 illustrates the phases of the (appropriately rescaled) oscillations in the quasi-stationary state in the same situation as above ($k = n = 4$). The observations corroborate

again (3.3).

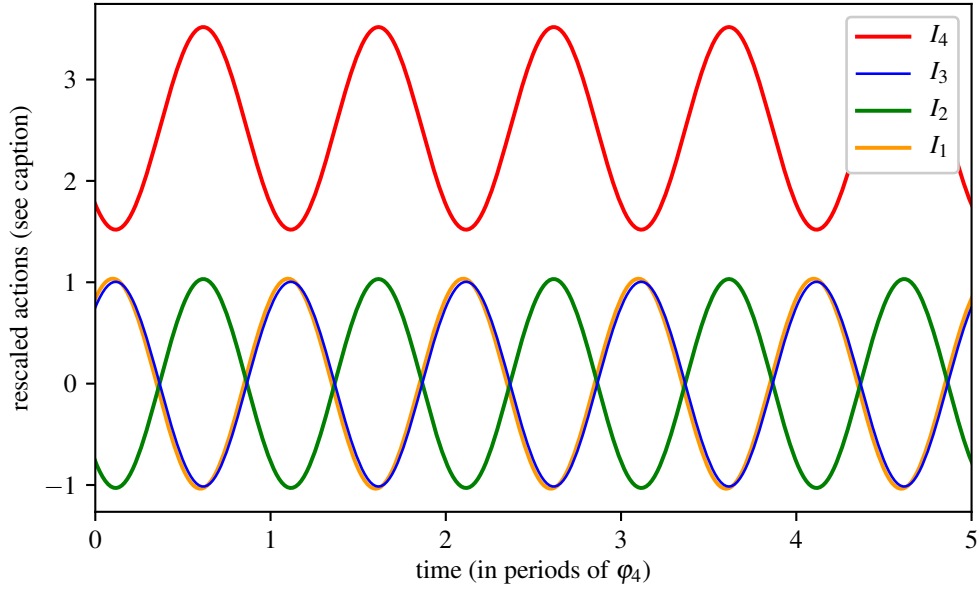


Figure 2. Oscillations in the quasi-stationary state $n = k = 4$, $\gamma = 1$ and $L = 10$. The amplitudes of I_i are rescaled by the predicted factors $L^{2|4-i|-1}$, except for I_4 (the fast rotor), which is displayed as $(I_4 - \langle I_4 \rangle) \cdot L$ and shifted upward for better visibility.

In Fig. 3, we show the results for the case $n = 6$ and $k = 4$. We observe that the amplitudes in the quasi-stationary state depend only on the absolute value of $i - k$, and are compatible with (3.3).

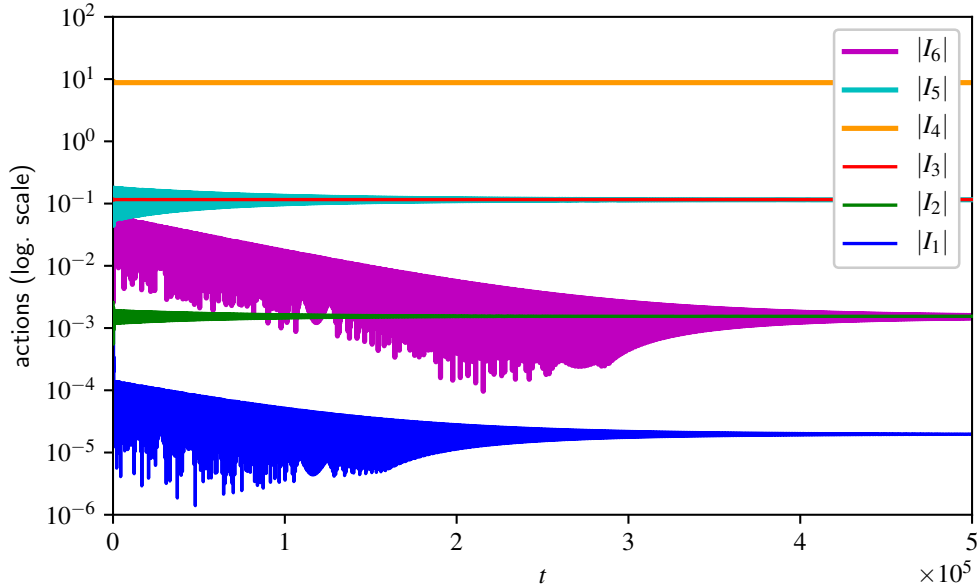


Figure 3. Maximum of $|I_i|$ over intervals of length $2\pi/L$ for $n = 6$, $k = 4$, $\gamma = 2$ and $L = 9$.

4. Notation and tools

We will consider analytic functions on a complex neighborhood of the phase space Ω .

Definition 4.1. For any $L, r, \sigma > 0$, we define a complex domain (in \mathbf{C}^{2n}) by

$$\mathcal{D}_{L,r,\sigma} = \{(\mathbf{I}, \boldsymbol{\varphi}) \in \mathbf{C}^{2n} : |I_i - L\delta_{i,k}| < rL, |\operatorname{Im}\varphi_i| < \sigma, i = 1, \dots, n\}.$$

and a norm on analytic, $\boldsymbol{\varphi}$ -periodic functions

$$\|f\|_{L,r,\sigma} = \sup_{\mathcal{D}_{L,r,\sigma}} |f(\mathbf{I}, \boldsymbol{\varphi})|. \quad (4.1)$$

We say that a $\boldsymbol{\varphi}$ -periodic function f is admissible if there exist $L_0, r, \sigma > 0$ such that for all $L \geq L_0$, f is analytic on $\mathcal{D}_{L,r,\sigma}$ and $\|f\|_{L,r,\sigma} < \infty$.

By assumption, $f^{(0)}$ is admissible (we assume throughout that Assumptions 2.1 and 2.2 are in force). In fact, $\|f^{(0)}\|_{L,r,\sigma}$ is finite for every r, σ , and independent of L .

Remark 4.2. We make the distinction between $\mathcal{D}_{L,r,\sigma}$ and $\mathcal{B}_{L,\varrho}$ clear:

- The set $\mathcal{D}_{L,r,\sigma}$ is complex, and the actions I_i , $i \neq k$ are allowed to grow with L . We will mostly work in terms of $\mathcal{D}_{L,r,\sigma}$ when discussing analytic properties of functions. In particular we need this scaling in order to obtain good bounds on the derivatives with respect to I_i , using Cauchy's theorem (see Lemma A.4).
- The set $\mathcal{B}_{L,\varrho}$ is real, and the actions I_i , $i \neq k$ absolutely bounded by ϱ , independently on L . This will be more convenient when working with the orbits of the system, since as stated in Theorem 2.4 (i), they remain for long times in such sets.

Remark 4.3. For all $\varrho, r, \sigma > 0$, we have $\mathcal{B}_{L,\varrho} \subset \mathcal{D}_{L,r,\sigma}$ as soon as $L \geq \varrho/r$.

4.1. Fourier series and resonance

Given an admissible function f , we write its Fourier series with respect to $\boldsymbol{\varphi}$ as

$$f(\mathbf{I}, \boldsymbol{\varphi}) = \sum_{\boldsymbol{\mu} \in \mathcal{N}} f_{\boldsymbol{\mu}}(\mathbf{I}) e^{i\boldsymbol{\mu} \cdot \boldsymbol{\varphi}}, \quad (4.2)$$

where $\mathcal{N} \subset \mathbf{Z}^n$. The Fourier series above is well-defined in $\mathcal{D}_{L,r,\sigma}$ for r, σ small enough and all L large enough.

Definition 4.4. Given a function f and its Fourier series (4.2), we define

$$\begin{aligned} \operatorname{supp}(\boldsymbol{\mu}) &= \{i : \mu_i \neq 0\}, \\ |\boldsymbol{\mu}| &= \max\{|\mu_i|, i = 1, \dots, n\}, \\ |\mathcal{N}| &= \sup\{|\boldsymbol{\mu}| : \boldsymbol{\mu} \in \mathcal{N}\}. \end{aligned}$$

4.2. Sets of interactions and the non-resonance condition

We will use the following notation:

Definition 4.5. We decompose any $\mathcal{N} \subset \mathbf{Z}^n$ as a disjoint union $\mathcal{N} = \mathcal{N}^{\text{NR}} \cup \mathcal{N}^{\text{R}}$:

$$\mathcal{N}^{\text{NR}} = \{\boldsymbol{\mu} \in \mathcal{N} : \mu_k \neq 0\}, \quad \mathcal{N}^{\text{R}} = \{\boldsymbol{\mu} \in \mathcal{N} : \mu_k = 0\}, \quad (4.3)$$

where NR stands for non-resonant interactions, and R stands for resonant interactions. Given a function f of the form (4.2), we decompose correspondingly

$$f = f^{\text{R}} + f^{\text{NR}}, \quad (4.4)$$

where

$$f^{\text{R}} = \sum_{\boldsymbol{\mu} \in \mathcal{N}^{\text{R}}} f_{\boldsymbol{\mu}}(\mathbf{I}) e^{i\boldsymbol{\mu} \cdot \boldsymbol{\varphi}}, \quad f^{\text{NR}} = \sum_{\boldsymbol{\mu} \in \mathcal{N}^{\text{NR}}} f_{\boldsymbol{\mu}}(\mathbf{I}) e^{i\boldsymbol{\mu} \cdot \boldsymbol{\varphi}}. \quad (4.5)$$

For example, we have the decomposition $f^{(0)} = f^{(0,\text{R})} + f^{(0,\text{NR})}$, where

$$\begin{aligned} f^{(0,\text{R})} &= \sum_{i \in \{1, \dots, n-1\} \setminus \{k-1, k\}} U_i(\varphi_{i+1} - \varphi_i), \\ f^{(0,\text{NR})} &= \sum_{i \in \{1, \dots, n-1\} \cap \{k-1, k\}} U_i(\varphi_{i+1} - \varphi_i). \end{aligned} \quad (4.6)$$

Note that resonant part of the interaction includes all terms in which any rotator couples to rotator k , and the non-resonant parts include any terms with no interaction with rotator k .

Remark 4.6. This notion of resonant and non-resonant is justified as follows. When I_k is very large and the other actions are small, we have the following approximate dynamics for short times:

$$\begin{aligned} \varphi_k(t) &\approx \varphi_k(0) + tI_k, \\ \varphi_i(t) &\approx \varphi_i(0). \end{aligned}$$

With respect to this dynamics, $e^{i\boldsymbol{\mu} \cdot \boldsymbol{\varphi}(t)}$ essentially oscillates with frequency $I_k \mu_k$ if $\mu_k \neq 0$, and is essentially constant if $\mu_k = 0$. Taking the average over one period of length $2\pi/I_k$, the non-resonant terms cancel out, while the resonant ones remain unchanged. Thus, one can say that f^{NR} is the oscillatory part of f , while f^{R} is the average part of f .

Lemma 4.7. Let $\boldsymbol{\mu} \in \mathbf{Z}^n$ be non-resonant (i.e., $\mu_k \neq 0$), and assume $r < 1/(2n|\boldsymbol{\mu}|)$. Then for any $(\mathbf{I}, \boldsymbol{\varphi}) \in \mathcal{D}_{L,r,\sigma}$ one has

$$|\mathbf{I} \cdot \boldsymbol{\mu}| > \frac{L}{2}.$$

Proof. This is a consequence of the following easy computation:

$$\begin{aligned} |\mathbf{I} \cdot \boldsymbol{\mu}| &\geq |I_k \mu_k| - \left| \sum_{i \neq k} I_i \mu_i \right| \geq |\mu_k| L(1-r) - |\boldsymbol{\mu}|(n-1)rL \\ &\geq L(1-nr|\boldsymbol{\mu}|). \end{aligned}$$

□

Conversely, note that if μ is resonant, $\mathbf{I} \cdot \mu$ may vanish on any $\mathcal{D}_{L,r,\sigma}$. That distinction will be crucial in the sequel, and we will need to check that all factors of $\mathbf{I} \cdot \mu$ appearing in the denominators of the functions we construct have μ non-resonant.

4.3. Lie series

We will eliminate the non-resonant terms by a Lie transformation, writing $H' = H \circ \Phi$, where Φ is the time-1 flow generated by the Hamiltonian vector field X_χ of a well-chosen Hamiltonian function χ .

To fix the notation, we use here the terminology of Pöschel [33]. He defines the canonical transformation Φ as

$$G \circ \Phi = \sum_{s=0}^{\infty} \frac{1}{s!} \text{ad}_\chi^s G, \quad (4.7)$$

with

$$\text{ad}_\chi^0 G = G, \quad \text{ad}_\chi^s G = \{\text{ad}_\chi^{s-1} G, \chi\}.$$

The Poisson bracket $\{\cdot, \cdot\}$ is defined as

$$\{F, G\} = \sum_{i=1}^n \left(\frac{\partial F}{\partial \varphi_i} \frac{\partial G}{\partial I_i} - \frac{\partial F}{\partial I_i} \frac{\partial G}{\partial \varphi_i} \right).$$

With a slight abuse of notation, we will write

$$G \circ \Phi \equiv e^\chi G,$$

where the right-hand side really means $e^{\text{ad}_\chi} G$ and is defined by (4.7). The transformation Φ is canonical, and its inverse is given by $G \circ \Phi^{-1} = e^{-\chi} G$.

The Hamiltonian evolution of some function F with respect to the Hamiltonian H is given by the Poisson bracket $\{F, H\}$. The evolution of the rotators in our model consist of two pieces: a Hamiltonian piece which we can express in terms of the Poisson bracket and the dissipative term that affects the first rotator. More explicitly, we have

$$\frac{d}{dt} I_i = \{I_i, H\} - \gamma I_1 \delta_{1,i}, \quad \frac{d}{dt} \varphi_i = \{\varphi_i, H\}.$$

We will transform the Hamiltonian part of the equations with the aid of the Lie transform method to almost completely decouple the k^{th} rotator from the remainder of the system. We will then have to examine carefully the effects of these transforms on the dissipative term.

4.4. Eliminating non-resonant interactions

We now explain in detail the main iteration of our process in an abstract setting. Assume we have a Hamiltonian of the form

$$H = h^{(0)} + f + g = h^{(0)} + f^{\text{NR}} + f^{\text{R}} + g,$$

where the decomposition (4.2) of f contains *finitely many* terms, and g is any function that we cannot, or do not need to remove. We would like to remove the resonant interactions f^{NR} from the new Hamiltonian by changing coordinates as in Sect. 4.3, with some well-chosen χ .

The transformation Φ transforms H into a new Hamiltonian H' given by

$$\begin{aligned} H' &= e^\chi H = h^{(0)} + f^{\text{NR}} + f^{\text{R}} + g + \{h^{(0)}, \chi\} + \{f + g, \chi\} \\ &\quad + \frac{1}{2!} \{\{H, \chi\}, \chi\} + \frac{1}{3!} \{\{\{H, \chi\}, \chi\}, \chi\} + \dots \\ &= h^{(0)} + f^{\text{NR}} + \{h^{(0)}, \chi\} + f^{\text{R}} + g \\ &\quad + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \text{ad}_\chi^\ell h^{(0)} + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \text{ad}_\chi^\ell (f + g). \end{aligned} \quad (4.8)$$

In order to remove the f^{NR} from (4.8), the idea is to choose χ so that

$$\{h^{(0)}, \chi\} = -f^{\text{NR}}. \quad (4.9)$$

Such a χ is constructed as follows.

Definition 4.8. For any function f as in (4.2), with \mathcal{N} finite, we let

$$Qf \equiv -i \sum_{\mu \in \mathcal{N}^{\text{NR}}} \frac{f_\mu(\mathbf{I})}{\mathbf{I} \cdot \mu} e^{i\mu \cdot \varphi}. \quad (4.10)$$

Since \mathcal{N}^{NR} is finite, we have by Lemma 4.7 that the denominators $\mathbf{I} \cdot \mu$ appearing here are well defined and larger than $L/2$ in absolute value on $\mathcal{D}_{L,r,\sigma}$ for some appropriate r, σ , and all large enough L . Thus, Qf is well defined on that domain. Moreover, since these denominators scale like L , we reasonably expect Qf to be “small” when L is large. This will be made precise later. We observe also that if f is real (i.e., $f_{-\mu} = \bar{f}_\mu$) then so is Qf . Finally, by construction,

$$\{h^{(0)}, Qf\} = - \sum_{i=1}^n I_i \partial_{\varphi_i} Qf = - \sum_{\mu \in \mathcal{N}^{\text{NR}}} f_\mu(\mathbf{I}) e^{i\mu \cdot \varphi} = -f^{\text{NR}}, \quad (4.11)$$

so that choosing

$$\chi = Qf$$

guarantees that (4.9) holds. As a consequence, (4.8) becomes

$$\begin{aligned} H' &= h^{(0)} + f^{\text{R}} + g + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \text{ad}_\chi^\ell h^{(0)} + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \text{ad}_\chi^\ell (f + g) \\ &= h^{(0)} + f^{\text{R}} + g + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \text{ad}_\chi^\ell \left(f + g - \frac{f^{\text{NR}}}{\ell + 1} \right). \end{aligned} \quad (4.12)$$

Thus, f^{NR} has indeed been removed from H' (while f^{R} remains untouched). The price to pay is the appearance of some new interactions in the form of the infinite series above. The idea is then to eliminate part of this new term by another canonical transformation, and to iterate this procedure as many times as needed.

We address now the natural question of whether this infinite series converges, and whether it is actually smaller than the interaction f^{NR} that we have removed.

4.5. Orders of magnitude

Definition 4.9. Let f be admissible, and let $s \in \mathbf{R}$. We say that

$$f = \widehat{\mathcal{O}}(L^s) \quad (4.13)$$

if there exist constants $r, \sigma, L_0, C > 0$ such that for all $L \geq L_0$,

$$\|f\|_{L,r,\sigma} \leq CL^s. \quad (4.14)$$

The following properties are proved in Appendix A, and will often be used without reference.

Lemma 4.10. Let $f = \widehat{\mathcal{O}}(L^s)$ and $g = \widehat{\mathcal{O}}(L^z)$. Then,

- (a) $f + g = \widehat{\mathcal{O}}(L^{\max(s,z)})$,
- (b) $fg = \widehat{\mathcal{O}}(L^{s+z})$,
- (c) $\partial_{\varphi_i} f = \widehat{\mathcal{O}}(L^s)$,
- (d) $\partial_{I_i} f = \widehat{\mathcal{O}}(L^{s-1})$,
- (e) $\{f, g\} = \widehat{\mathcal{O}}(L^{s+z-1})$,
- (f) $\text{ad}_g^\ell f = \widehat{\mathcal{O}}(L^{s+\ell(z-1)})$,
- (g) if $(a_\ell)_{\ell \geq 0}$ is a bounded sequence, then for all $0 \leq \ell_0 \leq \ell_1 \leq \infty$,

$$\sum_{\ell=\ell_0}^{\ell_1} \frac{a_\ell}{\ell!} \text{ad}_g^\ell f = \widehat{\mathcal{O}}(L^{s+\ell_0(z-1)}),$$

- (h) if f has the form (4.2) and $\tilde{\mathcal{N}} \subset \mathcal{N}$, then

$$\sum_{\mu \in \tilde{\mathcal{N}}} f_\mu(\mathbf{I}) e^{i\mu \cdot \varphi} = \widehat{\mathcal{O}}(L^s),$$

- (i) if in addition f contains finitely many Fourier modes, then

$$Qf = \widehat{\mathcal{O}}(L^{s-1}).$$

Proof. (a) and (b) are obvious. (c), (d) and (e) follow from Cauchy's theorem (see Lemma A.4 and Corollary A.5). (f) follows from iterating (e) and is proved in Lemma A.6. (g) is slightly more involved, as when $\ell_1 = \infty$ one must make sure that the domain does not shrink too much (see Proposition A.7). A proof of (h) is provided in Lemma A.2. Finally, (i) is proved in Proposition A.3: the idea that in the definition of Q , each term is divided by $\mu \cdot \mathbf{I}$, which scales like L in an appropriate domain by Lemma 4.7. \square

Each operation in Lemma 4.10 imposes further restrictions on the parameters r, σ, L_0 in (4.13) and (4.14) (see Appendix A for the details). This will, however, not be a problem since only a *finite number* of such operations will be needed.

As φ_i is not strictly speaking an admissible function (since it is not φ -periodic on \mathbf{C}^{2n}), we need a supplementary lemma, which basically says that φ_i behaves like an $\widehat{\mathcal{O}}(L^0)$.

Lemma 4.11. *Let $g = \widehat{\mathcal{O}}(L^z)$. Then*

$$\{\varphi_i, g\} = \widehat{\mathcal{O}}(L^{z-1}) ,$$

and for all $1 \leq \ell_0 \leq \ell_1 \leq \infty$,

$$\sum_{\ell=\ell_0}^{\ell_1} \frac{1}{\ell!} \text{ad}_g^\ell \varphi_i = \widehat{\mathcal{O}}(L^{\ell_0(z-1)}) .$$

Proof. The first statement follows from the fact that $\{\varphi_i, g\} = \partial_{I_i} g$ and Lemma 4.10 (d). As a consequence, we have

$$\sum_{\ell=\ell_0}^{\ell_1} \frac{1}{\ell!} \text{ad}_g^\ell \varphi_i = \sum_{\ell=\ell_0-1}^{\ell_1-1} \frac{1}{\ell!} \text{ad}_g^\ell \frac{\partial_{I_i} g}{\ell+1} ,$$

which by Lemma 4.10 (g) proves the second statement. \square

5. Inductive construction

In this section, we describe the successive steps in which several canonical transformations eliminate the non-resonant terms. Throughout, we fix a cutoff N_* . We will choose N_* big enough so that the error terms $R^{(j)}$ below are negligible powers of L . (We will see that $N_* = k$ is good enough.) In a first reading, the reader may ignore this cutoff and the corresponding remainders $R^{(j)}$ (formally taking $N_* = \infty$).

We do the first two iterations explicitly in order to gain some intuition, and then describe the general step.

5.1. The first canonical transformation

We start with the Hamiltonian $H^{(0)} = h^{(0)} + f^{(0)}$ of the form of (2.1), and decompose $f^{(0)}$ according to (4.4) and (4.5). This leads to

$$H^{(0)} = h^{(0)} + f^{(0,\text{R})} + f^{(0,\text{NR})} ,$$

where the explicit expression of $f^{(0,\text{R})}$ and $f^{(0,\text{NR})}$ was given in (4.6). Obviously, we have $f^{(0)} = \widehat{\mathcal{O}}(L^0)$. Following Sect. 4.4 (with $f = f^{(0)}$ and $g = 0$), we let

$$\chi^{(0)} = Q f^{(0)} .$$

We have $\{h^{(0)}, \chi^{(0)}\} = -f^{(0,\text{NR})}$, and by Lemma 4.10 (h),

$$\chi^{(0)} = \widehat{\mathcal{O}}(L^{-1}) . \tag{5.1}$$

By (4.12), we thus find that the non-resonant term $f^{(0,\text{NR})}$ is removed, and that

$$H^{(1)} \equiv e^{\chi^{(0)}} H^{(0)} = h^{(0)} + f^{(0,\text{R})} + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \text{ad}_{\chi^{(0)}}^\ell \left(f^{(0)} - \frac{f^{(0,\text{NR})}}{\ell+1} \right)$$

$$= h^{(0)} + f^{(0,\text{R})} + f^{(1)} + R^{(1)},$$

where

$$\begin{aligned} f^{(1)} &= \sum_{\ell=1}^{N_*-1} \frac{1}{\ell!} \text{ad}_{\chi^{(0)}}^\ell \left(f^{(0)} - \frac{f^{(0,\text{NR})}}{\ell+1} \right) = \widehat{\mathcal{O}}(L^{-2}), \\ R^{(1)} &= \sum_{\ell=N_*}^{\infty} \frac{1}{\ell!} \text{ad}_{\chi^{(0)}}^\ell \left(f^{(0)} - \frac{f^{(0,\text{NR})}}{\ell+1} \right) = \widehat{\mathcal{O}}(L^{-2N_*}). \end{aligned} \quad (5.2)$$

The estimates $\widehat{\mathcal{O}}(L^{-2})$ and $\widehat{\mathcal{O}}(L^{-2N_*})$ here come from Lemma 4.10 (g), (5.1), and the fact that $f^{(0)} = \widehat{\mathcal{O}}(L^0)$ (note that by Lemma 4.10 (h), also $f^{(0,\text{NR})} = \widehat{\mathcal{O}}(L^0)$).

We now pause to make a series of observations and comments.

First, we put only finitely many terms in $f^{(1)}$, and the rest of the infinite series goes into $R^{(1)}$. The reason for this is that at the next step, we want to remove the non-resonant part of $f^{(1)}$ by making another canonical transformation given by $\chi^{(1)} = Qf^{(1)}$, and for this we cannot have infinitely many terms in $f^{(1)}$. As mentioned above, N_* will be chosen so that the remainder $R^{(1)}$ is small enough.

Secondly, and this is the essential feature, the coupling between the site k and its neighbors, which was in $f^{(0,\text{NR})}$, has been removed. Actually, the only interactions now involving the site k appear in $f^{(1)}$ and $R^{(1)}$, so one can say that the site k has been *decoupled* up to order L^{-2} from the rest of the chain.

Next, we make

Observation 5.1. *The function $f^{(1)}$ depends only on the variables (φ_i, I_i) for $i \in \{k-2, \dots, k+2\}$. (We should actually write $\{\max(1, k-2), \dots, \min(k+2, n)\}$. To avoid burdening the notation, we will often omit to mention that the dependence is obviously restricted to the sites $\{1, \dots, n\}$.)*

This is seen as follows. By the structure of $f^{(0)}$ as a sum of two-body interactions, and by the definition of $\chi^{(0)}$, we have that $\chi^{(0)}$ depends only on the sites $k-1, k, k+1$. It is then easy to realize, using the structure of $f^{(0)}$ again, that

$$\left\{ f^{(0)} - \frac{f^{(0,\text{NR})}}{\ell+1}, \chi^{(0)} \right\} \quad (5.3)$$

depends on the sites $k-2, \dots, k+2$. In fact, the dependence is extended to the sites $k-2$ and $k+2$ due to the terms U_{k-2} and U_{k+1} in $f^{(0)}$: the Poisson bracket $\{U_{k-2}(\varphi_{k-1} - \varphi_{k-2}), \chi^{(0)}\}$ couples the sites k and $k-2$, and the Poisson bracket $\{U_{k+1}(\varphi_{k+2} - \varphi_{k+1}), \chi^{(0)}\}$ couples the sites k and $k+2$. Moreover, starting with (5.3) and taking more Poisson brackets with $\chi^{(0)}$ does not extend the dependence to more sites, which gives the observation above.

To summarize, at the end of this first canonical transformation, we have replaced the non-resonant interactions $f^{(0,\text{NR})}$ with some smaller term $f^{(1)} = \widehat{\mathcal{O}}(L^{-2})$ and a very small remainder $R^{(1)} = \widehat{\mathcal{O}}(L^{-2N_*})$. The site k is then decoupled up to order L^{-2} , but the potential $f^{(1)}$ makes it interact with its next-to-nearest neighbors (whereas $H^{(0)}$ only featured nearest-neighbors interactions).

5.2. The second canonical transformation

We first decompose

$$\begin{aligned} H^{(1)} &= h^{(0)} + f^{(0,R)} + f^{(1)} + R^{(1)} \\ &= h^{(0)} + f^{(0,R)} + f^{(1,NR)} + f^{(1,R)} + R^{(1)}. \end{aligned}$$

We now want to remove $f^{(1,NR)}$. Following again the method of Sect. 4.4, this time with $f = f^{(1)}$ and $g = R^{(1)} + f^{(0,R)}$, we define

$$\chi^{(1)} = Qf^{(1)},$$

so that $\{h^{(0)}, \chi^{(1)}\} = -f^{(1,NR)}$ and $\chi^{(1)} = \widehat{\mathcal{O}}(L^{-3})$. It is easy to see that (4.12) can be rearranged as

$$H^{(2)} \equiv e^{\chi^{(1)}} H^{(1)} = h^{(0)} + f^{(0,R)} + f^{(1,R)} + f^{(2)} + R^{(2)},$$

with

$$\begin{aligned} f^{(2)} &= \sum_{\ell=1}^{N_*-1} \frac{1}{\ell!} \text{ad}_{\chi^{(1)}}^\ell \left(f^{(0,R)} + f^{(1)} - \frac{f^{(1,NR)}}{\ell+1} \right) = \widehat{\mathcal{O}}(L^{-4}), \\ R^{(2)} &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \text{ad}_{\chi^{(1)}}^\ell R^{(1)} + \sum_{\ell=N_*}^{\infty} \frac{1}{\ell!} \text{ad}_{\chi^{(1)}}^\ell \left(f^{(0,R)} + f^{(1)} - \frac{f^{(1,NR)}}{\ell+1} \right) = \widehat{\mathcal{O}}(L^{-2N_*}), \end{aligned} \tag{5.4}$$

where we have used Lemma 4.10 (g).

The potentials $f^{(0,R)}$ and $f^{(1,R)}$ do not involve φ_k by definition. Thus, since only $f^{(2)}$ and $R^{(2)}$ involve φ_k , we observe that the rotor k is now decoupled up to order L^{-4} .

Since $f^{(1)}$, and hence also $\chi^{(1)}$, involve only the sites $k-2, \dots, k+2$, we observe that

$$\left\{ f^{(0,R)} + f^{(1)} - \frac{f^{(1,NR)}}{\ell+1}, \chi^{(1)} \right\} \tag{5.5}$$

depends only on the variables of the sites $k-3, \dots, k+3$. In fact, the support is only extended due to $f^{(0,R)}$: the Poisson brackets $\{U_{k-3}(\varphi_{k-2} - \varphi_{k-3}), \chi^{(1)}\}$ and $\{U_{k+2}(\varphi_{k+3} - \varphi_{k+2}), \chi^{(1)}\}$ couple the site k to the sites $k+3$ and $k-3$. Taking further Poisson brackets of (5.5) with $\chi^{(1)}$ does not extend the range of the interactions, so we obtain

Observation 5.2. *The function $f^{(2)}$ involves only the sites $k-3, \dots, k+3$.*

5.3. Canonical transformations for $j > 1$

We are now ready for the inductive step. We start with

$$H^{(j-1)} = h^{(0)} + \sum_{m=0}^{j-2} f^{(m,R)} + f^{(j-1)} + R^{(j-1)},$$

where

$$\begin{aligned} f^{(j-1)} &= \widehat{\mathcal{O}}(L^{-2(j-1)}) , \\ R^{(j-1)} &= \widehat{\mathcal{O}}(L^{-2N_*}) , \end{aligned} \quad (5.6)$$

and where $f^{(j-1)}$ depends only on the sites $k-j, \dots, k+j$.

We decompose again

$$f^{(j-1)} = f^{(j-1, \text{NR})} + f^{(j-1, \text{R})} ,$$

and in order to remove $f^{(j-1, \text{NR})}$, we follow Sect. 4.4 with

$$f = f^{(j-1)} , \quad g = \sum_{m=0}^{j-2} f^{(m, \text{R})} + R^{(j-1)} .$$

We thus define

$$\chi^{(j-1)} = Q f^{(j-1)} = \widehat{\mathcal{O}}(L^{1-2j}) , \quad (5.7)$$

so that $\{h^{(0)}, \chi^{(j-1)}\} = -f^{(j-1, \text{NR})}$. This yields, by rearranging the terms of (4.12),

$$H^{(j)} \equiv e^{\chi^{(j-1)}} H^{(j-1)} = h^{(0)} + \sum_{m=0}^{j-1} f^{(m, \text{R})} + f^{(j)} + R^{(j)} , \quad (5.8)$$

with

$$\begin{aligned} f^{(j)} &= \sum_{\ell=1}^{N_*-1} \frac{1}{\ell!} \text{ad}_{\chi^{(j-1)}}^\ell \left(\sum_{m=0}^{j-2} f^{(m, \text{R})} + f^{(j-1)} - \frac{f^{(j-1, \text{NR})}}{\ell+1} \right) , \\ R^{(j)} &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \text{ad}_{\chi^{(j-1)}}^\ell R^{(j-1)} + \sum_{\ell=N_*}^{\infty} \frac{1}{\ell!} \text{ad}_{\chi^{(j-1)}}^\ell \left(\sum_{m=0}^{j-2} f^{(m, \text{R})} + f^{(j-1)} - \frac{f^{(j-1, \text{NR})}}{\ell+1} \right) . \end{aligned} \quad (5.9)$$

At this point, we have an inductive definition of $f^{(j)}$ and $R^{(j)}$ (one can set $R^{(0)} = 0$). By Lemma 4.10 (g) and (5.6), we find

$$\begin{aligned} f^{(j)} &= \widehat{\mathcal{O}}(L^{-2j}) , \\ R^{(j)} &= \widehat{\mathcal{O}}(L^{-2N_*}) . \end{aligned} \quad (5.10)$$

Moreover, since $\chi^{(j-1)}$ only involves the variables $k-j, \dots, k+j$, we find that

$$\left\{ \sum_{m=0}^{j-2} f^{(m, \text{R})} + f^{(j-1)} - \frac{f^{(j-1, \text{NR})}}{\ell+1}, \chi^{(j-1)} \right\} \quad (5.11)$$

depends only on the sites $k-j-1, \dots, k+j+1$. Indeed, the first argument of the Poisson brackets depends only on $k-j, \dots, k+j$, except for the term $f^{(0, \text{R})}$, which once again extends the range of the interactions by one due to $U_{k-j-1}(\varphi_{k-j} - \varphi_{k-j-1})$ and $U_{k+j}(\varphi_{k+j+1} - \varphi_{k+j})$. Taking further Poisson brackets of (5.11) with $\chi^{(j-1)}$ does not extend the dependence to new variables, so that we finally have

Observation 5.3. For $j \geq 1$, the function $f^{(j)}$ depends only on the sites $\max(1, k - j - 1), \dots, \min(k + j + 1, n)$.

(Note that we have to exclude the case $j = 0$ from this statement, since the function $f^{(0)}$ involves all of the angles.)

The observation above as well as the orders in (5.9) are illustrated in Table 1 in the case $n = 9$ and $k = 6$.

	1	2	3	4	5	6	7	8	9	
$f^{(0)}$	•	•	•	•	•	•	•	•	•	$\widehat{\mathcal{O}}(L^0)$
$\chi^{(0)}$					•	•	•			$\widehat{\mathcal{O}}(L^{-1})$
$f^{(1)}$				•	•	•	•	•		$\widehat{\mathcal{O}}(L^{-2})$
$\chi^{(1)}$				•	•	•	•	•		$\widehat{\mathcal{O}}(L^{-3})$
$f^{(2)}$			•	•	•	•	•	•	•	$\widehat{\mathcal{O}}(L^{-4})$
$\chi^{(2)}$			•	•	•	•	•	•	•	$\widehat{\mathcal{O}}(L^{-5})$
$f^{(3)}$		•	•	•	•	•	•	•	•	$\widehat{\mathcal{O}}(L^{-6})$
$\chi^{(3)}$		•	•	•	•	•	•	•	•	$\widehat{\mathcal{O}}(L^{-7})$
$f^{(4)}$	•	•	•	•	•	•	•	•	•	$\widehat{\mathcal{O}}(L^{-8})$
$\chi^{(4)}$	•	•	•	•	•	•	•	•	•	$\widehat{\mathcal{O}}(L^{-9})$

Table 1. Illustration of the iterative construction in the case $n = 9$ and $k = 6$. The bullets indicate the dependence on the sites.

6. Dynamics after $k - 1$ iterations

We stop the process above after $k - 1$ iterations. Defining

$$\widetilde{H} \equiv H^{(k-1)} = h^{(0)} + \sum_{m=0}^{k-2} f^{(m,R)} + f^{(k-1)} + R^{(k-1)} \quad (6.1)$$

and letting

$$N_* = k ,$$

we find the estimates

$$\begin{aligned} \sum_{m=0}^{k-2} f^{(m,R)} &= \widehat{\mathcal{O}}(L^0) , \\ f^{(k-1)} &= \widehat{\mathcal{O}}(L^{2-2k}) , \\ R^{(k-1)} &= \widehat{\mathcal{O}}(L^{-2k}) . \end{aligned}$$

Remark 6.1. In \widetilde{H} , the site k is decoupled from its neighbors up to order L^{2-2k} . However, if $k < n$, this does not imply that the subsystems $\{(\widetilde{I}_i, \widetilde{\varphi}_i) : i < k\}$ and $\{(\widetilde{I}_i, \widetilde{\varphi}_i) : i > k\}$ are decoupled from each other up to order L^{2-2k} . Indeed, $f^{(1,R)}$ contains terms of order L^{-2} (i.e., of rather “low” order) coupling the sites $k - 1$ and $k + 1$.

Tracing back the different steps, we have

$$\begin{aligned}\tilde{H} &= e^{\chi^{(k-2)}} e^{\chi^{(k-3)}} \dots e^{\chi^{(0)}} H \\ &= H \circ \Psi ,\end{aligned}$$

where Ψ is the composition of the time-one flow of the Hamiltonian fields of $\chi^{(0)}, \chi^{(1)}, \dots, \chi^{(k-2)}$. The coordinates $x = (\mathbf{I}, \boldsymbol{\varphi})$ are then transformed into new coordinates $\tilde{x} = (\tilde{\mathbf{I}}, \tilde{\boldsymbol{\varphi}}) = \Psi^{-1}(x)$, given by

$$\tilde{x}_a = \Psi_a^{-1}(x) \equiv e^{-\chi^{(0)}} e^{-\chi^{(1)}} \dots e^{-\chi^{(k-2)}} x_a , \quad (6.2)$$

where the argument of the $\chi^{(i)}$ is x . Here and in the sequel, x_a denotes any component of $(\mathbf{I}, \boldsymbol{\varphi})$. Conversely,

$$x_a = \Psi_a(\tilde{x}) = e^{\chi^{(k-2)}} e^{\chi^{(k-3)}} \dots e^{\chi^{(0)}} \tilde{x}_a ,$$

where the argument of the $\chi^{(i)}$ is now \tilde{x} , and the Poisson brackets are taken with respect to the \tilde{x} , that is according to the rules $\{\tilde{\varphi}_i, \tilde{\varphi}_j\} = \{\tilde{I}_i, \tilde{I}_j\} = 0$ and $\{\tilde{\varphi}_i, \tilde{I}_j\} = \delta_{i,j}$.

Obviously, we have

$$\frac{d}{dt} \tilde{x}_a = \{\tilde{x}_a, \tilde{H}\} - \gamma I_1 \partial_{I_1} \tilde{x}_a , \quad (6.3)$$

where again the Poisson brackets with respect to \tilde{x} are used, and the argument of \tilde{H} is \tilde{x} .

We now state and prove a series of technical results.

First, using recursively Lemma 4.10 (g), the fact that all the $\chi^{(j)}$ are at most $\hat{\mathcal{O}}(L^{-1})$, and the fact that $I_i = \hat{\mathcal{O}}(L^1)$, we have

$$\begin{aligned}\tilde{I}_i &= e^{-\chi^{(0)}} \dots e^{-\chi^{(k-2)}} I_i \\ &= I_i + \hat{\mathcal{O}}(L^{-1}) .\end{aligned} \quad (6.4)$$

Similarly, using Lemma 4.11 we find that

$$\begin{aligned}\tilde{\varphi}_i &= e^{-\chi^{(0)}} \dots e^{-\chi^{(k-2)}} \varphi_i \\ &= \varphi_i + \hat{\mathcal{O}}(L^{-2}) .\end{aligned} \quad (6.5)$$

Remark 6.2. The powers of L in (6.4) and (6.5) are not optimal. Recalling that $\chi^{(j)} = Qf^{(j)}$ depends only on the sites $k-j-1, \dots, k+j+1$, and that $\chi^{(j)} = \hat{\mathcal{O}}(L^{-2j-1})$, the interested reader can check that for $i \neq k$ we actually have $\tilde{I}_i - I_i = \hat{\mathcal{O}}(L^{1-2|k-i|})$ and $\tilde{\varphi}_i - \varphi_i = \hat{\mathcal{O}}(L^{-2|k-i|})$ (see Table 2). We shall not need the full extent of this result.

The following lemma and corollary guarantee that the domains from Definition 4.1 and the $\hat{\mathcal{O}}$ -notation are sufficiently stable under our change of coordinates.

Lemma 6.3. For all $r, \sigma, \varrho > 0$ small enough, and all $r' < r, \sigma' < \sigma, \varrho' < \varrho$, the following holds: for all L large enough,

$$\Psi^\sharp(\mathcal{D}_{L,r',\sigma'}) \subset \mathcal{D}_{L,r,\sigma} , \quad (6.6)$$

$$\Psi^\sharp(\mathcal{B}_{L,\varrho'}) \subset \mathcal{B}_{L,\varrho} , \quad (6.7)$$

where Ψ^\sharp stands for either Ψ^{-1} or Ψ .

i	Order	
	$\tilde{I}_i - I_i$	$\tilde{\varphi}_i - \varphi_i$
1	L^{3-2k}	L^{2-2k}
2	L^{5-2k}	L^{4-2k}
\dots	\dots	\dots
$k-2$	L^{-3}	L^{-4}
$k-1$	L^{-1}	L^{-2}
k	L^{-1}	L^{-2}
$k+1$	L^{-1}	L^{-2}
$k+2$	L^{-3}	L^{-4}
\dots	\dots	\dots

Table 2. An overview of the relevant powers of L in Remark 6.2.

Proof. By (6.4) and (6.5), we can find some $r_0, \sigma_0, L_0, C > 0$ such that for all $L \geq L_0$ and all $x \in \mathcal{D}_{L,r_0,\sigma_0}$, we have

$$\|\Psi^{-1}(x) - x\| \leq \frac{C}{L}.$$

Thus, for any $0 < \varepsilon < \varepsilon' < \varepsilon''$, we have some $L'_0 \geq L_0$ such that for all $L \geq L'_0$ and all $x \in \mathcal{D}_{L,r_0,\sigma_0}$, the following inclusions of complex balls holds:

$$B(x, \varepsilon) \subset \Psi^{-1}(B(x, \varepsilon')) \subset B(x, \varepsilon''). \quad (6.8)$$

The second inclusion immediately proves that $\Psi^{-1}(\mathcal{D}_{L,r',\sigma'}) \subset \mathcal{D}_{L,r,\sigma}$ for large enough L , provided that $r < r_0$ and $\sigma < \sigma_0$. Under the same conditions, the first inclusion in (6.8), which also reads $\Psi(B(x, \varepsilon)) \subset B(x, \varepsilon')$, proves that $\Psi(\mathcal{D}_{L,r',\sigma'}) \subset \mathcal{D}_{L,r,\sigma}$.

Next, (6.7) follows in a similar manner, using Remark 4.3 and the fact that Ψ maps real points to real points, so that a real equivalent of (6.8) holds. \square

The lemma above and the definition of $\widehat{\mathcal{O}}(L^s)$ immediately imply

Corollary 6.4. *If $f = \widehat{\mathcal{O}}(L^s)$, then*

$$f \circ \Psi^\sharp = \widehat{\mathcal{O}}(L^s), \quad (6.9)$$

where Ψ^\sharp stands for either Ψ^{-1} or Ψ .

Thanks to this corollary, we will not need to specify whether the $\widehat{\mathcal{O}}$ are to be expressed in terms of x or \tilde{x} . Note that the corollary can also be viewed as a consequence of Lemma 4.10 (g) and the fact that $f \circ \Psi = e^{\chi^{(k-2)}} e^{\chi^{(k-3)}} \dots e^{\chi^{(0)}} f$ and $f \circ \Psi^{-1} = e^{-\chi^{(0)}} e^{-\chi^{(1)}} \dots e^{-\chi^{(k-2)}} f$.

A series of definitions and technical results is necessary in order to understand the dynamics of \tilde{x} .

Definition 6.5. Let $f = \widehat{\mathcal{O}}(L^s)$. We say that $f = \widehat{\mathcal{O}}_{\text{R}}(L^s)$ if the Fourier series (4.2) contains only resonant terms (i.e., $f = f^{\text{R}}$), and similarly we say that $f = \widehat{\mathcal{O}}_{\text{NR}}(L^s)$ if the Fourier series contains only non-resonant terms (i.e., $f = f^{\text{NR}}$). Moreover, we denote by $\widehat{\mathcal{O}}^{\text{fin}}$, $\widehat{\mathcal{O}}_{\text{NR}}^{\text{fin}}$ and $\widehat{\mathcal{O}}_{\text{R}}^{\text{fin}}$ the remainders whose Fourier series (4.2) contains only finitely many Fourier modes.

Since $f^{(m,\text{R})} = \widehat{\mathcal{O}}_{\text{R}}^{\text{fin}}(L^{-2m})$, observe that (6.1) reads

$$\widetilde{H}(\widetilde{x}) = \sum_{i=1}^n \frac{\widetilde{I}_i^2}{2} + f^{(0,\text{R})}(\widetilde{\varphi}) + \widehat{\mathcal{O}}_{\text{R}}^{\text{fin}}(L^{-2}) + \widehat{\mathcal{O}}(L^{2-2k}). \quad (6.10)$$

We observe also that by (5.7) and the definition of $\chi^{(j)}$,

$$\chi^{(j)} = \widehat{\mathcal{O}}_{\text{NR}}^{\text{fin}}(L^{-2j-1}).$$

Lemma 6.6. We have

$$I_1 = \widetilde{I}_1 + P_1(\widetilde{x}), \quad \text{with} \quad P_1 = \widehat{\mathcal{O}}(L^{3-2k}).$$

More precisely, we have the decomposition

$$P_1 = -\partial_{\varphi_1} \chi^{(k-2)} + \widehat{\mathcal{O}}(L^{5-4k}). \quad (6.11)$$

Proof. Since the only $\chi^{(j)}$ that involves the site 1 is $\chi^{(k-2)}$, we find

$$I_1 = e^{\chi^{(k-2)}} \dots e^{\chi^{(0)}} \widetilde{I}_1 = e^{\chi^{(k-2)}} \widetilde{I}_1, \quad (6.12)$$

where it is understood that the argument of $\chi^{(j)}$ is \widetilde{x} . Since $\chi^{(k-2)} = \widehat{\mathcal{O}}(L^{3-2k})$, both statements immediately follow. \square

We will need bounds on the time evolution of \widetilde{x} . Because of the changes of variables, the dissipation no longer acts only on the first rotator. To compute the effect of dissipation on each \widetilde{x}_a , we need to understand how they depend on the original I_1 , onto which the dissipation acts.

Lemma 6.7. For $i = 1, 2, \dots, n$, we have

$$\begin{aligned} \frac{\partial \widetilde{I}_i}{\partial I_1} &= \delta_{1,i} + \widehat{\mathcal{O}}(L^{2-2k}), \\ \frac{\partial \widetilde{\varphi}_i}{\partial I_1} &= \widehat{\mathcal{O}}(L^{1-2k}). \end{aligned} \quad (6.13)$$

Proof. Since only $\chi^{(k-2)}$ involves the site 1, and $\chi^{(k-2)} = \widehat{\mathcal{O}}(L^{3-2k})$, we observe that (6.2) leads to

$$\begin{aligned} \frac{\partial \widetilde{I}_i}{\partial I_1} &= e^{-\chi^{(0)}} \dots e^{-\chi^{(k-3)}} \frac{\partial}{\partial I_1} \left(e^{-\chi^{(k-2)}} I_i \right) \\ &= e^{-\chi^{(0)}} \dots e^{-\chi^{(k-3)}} \frac{\partial}{\partial I_1} [I_i + \widehat{\mathcal{O}}(L^{3-2k})] \\ &= e^{-\chi^{(0)}} \dots e^{-\chi^{(k-3)}} [\delta_{1,i} + \widehat{\mathcal{O}}(L^{2-2k})] = \delta_{1,i} + \widehat{\mathcal{O}}(L^{2-2k}). \end{aligned}$$

(Recall that the Poisson bracket of $\delta_{1,j}$ and any function is zero.) For $\tilde{\varphi}_i$ we find similarly, using Lemma 4.11, that

$$\begin{aligned}\frac{\partial \tilde{\varphi}_i}{\partial I_1} &= e^{-\chi^{(0)}} \dots e^{-\chi^{(k-3)}} \frac{\partial}{\partial I_1} \left(e^{-\chi^{(k-2)}} \varphi_i \right) \\ &= e^{-\chi^{(0)}} \dots e^{-\chi^{(k-3)}} \frac{\partial}{\partial I_1} [\varphi_i + \hat{\mathcal{O}}(L^{2-2k})] \\ &= e^{-\chi^{(0)}} \dots e^{-\chi^{(k-3)}} [0 + \hat{\mathcal{O}}(L^{1-2k})] = \hat{\mathcal{O}}(L^{1-2k}),\end{aligned}$$

which completes the proof. \square

Lemma 6.8. *We have:*

$$\begin{aligned}\frac{d}{dt} \tilde{I}_i &= -\frac{\partial f^{(0,R)}}{\partial \varphi_i}(\tilde{\varphi}) + (1 - \delta_{i,k}) \hat{\mathcal{O}}_R^{\text{fin}}(L^{-2}) + \hat{\mathcal{O}}(L^{3-2k}) - \gamma \delta_{1,i} \tilde{I}_1, \\ \frac{d}{dt} \tilde{\varphi}_i &= \tilde{I}_i + \hat{\mathcal{O}}_R^{\text{fin}}(L^{-3}) + \hat{\mathcal{O}}(L^{2-2k}).\end{aligned}\tag{6.14}$$

Proof. We have

$$\frac{d}{dt} \tilde{I}_i = \{\tilde{I}_i, \tilde{H}\} - \gamma I_1 \frac{\partial}{\partial I_1} \tilde{I}_i.$$

First we note that by (6.10),

$$\{\tilde{I}_i, \tilde{H}\} = -\frac{\partial f^{(0,R)}}{\partial \varphi_i}(\tilde{\varphi}) + (1 - \delta_{i,k}) \hat{\mathcal{O}}_R^{\text{fin}}(L^{-2}) + \hat{\mathcal{O}}(L^{2-2k}),$$

where the $(1 - \delta_{i,k})$ comes from the fact that $\partial_{\varphi_k} \hat{\mathcal{O}}_R^{\text{fin}}(L^{-2}) = 0$, since the remainder is resonant and hence independent of φ_k . Moreover, we get by (6.13) and Lemma 6.6 that

$$I_1 \frac{\partial}{\partial I_1} \tilde{I}_i = (\tilde{I}_1 + \hat{\mathcal{O}}(L^{3-2k}))(\delta_{1,i} + \hat{\mathcal{O}}(L^{2-2k})) = \tilde{I}_1 \delta_{1,i} + \hat{\mathcal{O}}(L^{3-2k}).$$

This completes the proof of the first line of (6.14).

In the same way,

$$\frac{d}{dt} \tilde{\varphi}_i = \{\tilde{\varphi}_i, \tilde{H}\} - \gamma I_1 \frac{\partial}{\partial I_1} \tilde{\varphi}_i.$$

Now, using again (6.10) leads to

$$\{\tilde{\varphi}_i, \tilde{H}\} = \tilde{I}_i + \hat{\mathcal{O}}_R^{\text{fin}}(L^{-3}) + \hat{\mathcal{O}}(L^{1-2k}).$$

Since by Lemma 6.6 and (6.13),

$$I_1 \frac{\partial}{\partial I_1} \tilde{\varphi}_i = (\tilde{I}_1 + \hat{\mathcal{O}}(L^{3-2k})) \hat{\mathcal{O}}(L^{1-2k}) = \hat{\mathcal{O}}(L^{2-2k}),$$

the proof of the second line of (6.14) is complete. \square

Note that since $f^{(0,R)}$ does not involve φ_k by construction, (6.14) implies that

$$\frac{d}{dt} \tilde{I}_k = \hat{\mathcal{O}}(L^{3-2k}).\tag{6.15}$$

6.1. Long-time stability

In the sequel, $\alpha > 0$ and $T = \alpha L^{2k-3}$ are as in Theorem 2.4. We prove a statement slightly stronger than part (i) of Theorem 2.4, since it applies to both $x(t)$ and $\tilde{x}(t)$ (such a generalization will be needed to prove part (ii) of Theorem 2.4).

Proposition 6.9. *Let $\varrho > 0$. Then, there exist constants $L_0, \varrho^* > 0$ such that for all $L \geq L_0$ and all initial conditions $x(0) \in \mathcal{B}_{L,\varrho}$ we have both*

$$x(t) \in \mathcal{B}_{L,\varrho^*}, \quad \text{and} \quad \tilde{x}(t) \in \mathcal{B}_{L,\varrho^*}, \quad 0 \leq t \leq T.$$

Proof. We recall that since we consider real initial conditions $x \in \mathcal{B}_{L,\varrho}$, then both $x(t)$ and $\tilde{x}(t)$ remain real for as long as they are defined. In particular, we can determine whether the orbit remains in $\mathcal{B}_{L,\varrho^*}$ by just tracking the behavior of the action variables.

We define

$$\tilde{H}_* = \tilde{H} - \frac{\tilde{I}_k^2}{2} = \sum_{i \neq k} \frac{\tilde{I}_i^2}{2} + \hat{\mathcal{O}}(L^0). \quad (6.16)$$

Since $\frac{d}{dt} \tilde{H} = -\gamma I_1^2$, we have by (6.3)

$$\begin{aligned} \frac{d}{dt} \tilde{H}_* &= -\frac{d}{dt} \frac{\tilde{I}_k^2}{2} - \gamma I_1^2 = -\tilde{I}_k \left(\frac{\partial \tilde{H}}{\partial \varphi_k}(\tilde{x}) - \gamma I_1 \frac{\partial \tilde{I}_k}{\partial I_1} \right) - \gamma I_1^2 \\ &\leq -\tilde{I}_k \frac{\partial \tilde{H}}{\partial \varphi_k}(\tilde{x}) + \frac{\gamma}{4} \left(\tilde{I}_k \frac{\partial \tilde{I}_k}{\partial I_1} \right)^2 \\ &= \tilde{I}_k \hat{\mathcal{O}}(L^{2-2k}) + \tilde{I}_k^2 \hat{\mathcal{O}}(L^{4-4k}) = \hat{\mathcal{O}}(L^{3-2k}), \end{aligned} \quad (6.17)$$

where the second line follows from Young's inequality, and the third line follows from Lemma 6.7. (Note that the $\hat{\mathcal{O}}(L^{2-2k})$ in the last line comes from the fact that in (6.10), the resonant terms do not depend on φ_k by definition.)

By (6.16), (6.17) and (6.15), we obtain that there exist $r, \sigma, C > 0$ such that for all large enough L , we have on the set $\mathcal{D}_{L,r,\sigma}$,

$$\begin{aligned} 0 &\leq \sum_{i \neq k} \frac{\tilde{I}_i^2}{2} \leq \tilde{H}_* + C, \\ \frac{d}{dt} \tilde{H}_* &\leq CL^{3-2k}, \\ \left| \frac{d}{dt} \tilde{I}_k \right| &\leq CL^{3-2k}. \end{aligned} \quad (6.18)$$

Now, provided L is large enough, we have $\tilde{x}(0) \in \mathcal{B}_{L,2\varrho}$ by Lemma 6.3. Possibly increasing the value of C , we further require L to be large enough so that

$$\sup_{x \in \mathcal{B}_{L,2\varrho}} \tilde{H}_*(x) \leq C,$$

which is possible by Remark 4.3 and the definition of \tilde{H}_* . Choose now

$$\varrho^* > 2 \max(2\varrho + \alpha C, \sqrt{2(2 + \alpha)C}) .$$

By further restricting the allowed values of L , we have by Remark 4.3 that $\mathcal{B}_{L,\varrho^*} \subset \mathcal{D}_{L,r,\sigma}$. Consider now the first exit time

$$t^* = \inf\{t \geq 0 : \tilde{x}(t) \notin \mathcal{B}_{L,\varrho^*}\} ,$$

with the convention $\inf\{\emptyset\} = \infty$. For all $t \leq t^*$ the estimates in (6.18) hold, and thus for all $t \leq \min(T, t^*)$ we have

$$|\tilde{I}_k(t) - L| \leq |\tilde{I}_k(0) - L| + tCL^{3-2k} \leq 2\varrho + \alpha C < \frac{\varrho^*}{2} ,$$

and

$$\tilde{H}_*(\tilde{x}(t)) \leq \tilde{H}_*(\tilde{x}(0)) + tCL^{3-2k} \leq (1 + \alpha)C ,$$

which further implies that for all $i \neq k$,

$$|\tilde{I}_i(t)| \leq \sqrt{2(\tilde{H}_*(\tilde{x}(t)) + C)} \leq \sqrt{2(2 + \alpha)C} < \frac{\varrho^*}{2} .$$

This implies that for all $t \leq \min(T, t^*)$ we have $\tilde{x}(t) \in \mathcal{B}_{L,\varrho^*/2}$. By continuity of $\tilde{x}(t)$ with respect to t , this implies that $t^* > T$, so that actually $\tilde{x}(t) \in \mathcal{B}_{L,\varrho^*/2}$ for all $t \leq T$. But then by Lemma 6.3, we conclude that for all large enough L and all $t \leq T$, $x(t) \in \mathcal{B}_{L,\varrho^*}$. This completes the proof. \square

7. Estimating the dissipation

We now fix once and for all $\varrho, \alpha > 0$ as in Theorem 2.4. We fix also ϱ^* as in Proposition 6.9, so that for all large enough L , and all $x(0) \in \mathcal{B}_{L,\varrho}$,

$$x(t) \in \mathcal{B}_{L,\varrho^*}, \quad \tilde{x}(t) \in \mathcal{B}_{L,\varrho^*}, \quad 0 \leq t \leq T \equiv \alpha L^{2k-3} . \quad (7.1)$$

Convention. In the sequel, and without further mention, we consider only initial conditions $x(0) \in \mathcal{B}_{L,\varrho}$, where L is large enough so that (7.1) holds.

We will at several (but finitely many) occasions increase the lower bound on the allowed values for L . With this in mind, we now introduce a weaker notion of \mathcal{O} , which we will use alongside the $\hat{\mathcal{O}}$ introduced in Definition 4.9 (recall also Remark 4.2).

Definition 7.1. Given a function f defined on $\mathcal{B}_{L,\varrho^*}$ for all large enough L , we say that $f = \mathcal{O}(L^s)$ if there exists a constant c such that for all large enough L , we have $\sup_{x \in \mathcal{B}_{L,\varrho^*}} |f(x)| \leq cL^s$. The notations R, NR and fin bare the same meaning as in Definition 6.5.

We shall sometimes also write $\mathcal{O}(T^r L^s)$, which means $\mathcal{O}(L^{s+r(2k-3)})$ since $T = \alpha L^{2k-3}$. Moreover, with (7.1) in mind, we extend this notation to functionals of the trajectory: given $f = \mathcal{O}(L^s)$, we write for example $f(x(t)) = \mathcal{O}(L^s)$ when $t \leq T$, or $\int_0^T f(x(t)) dt = \mathcal{O}(TL^s)$.

By Remark 4.3, any f that is $\widehat{\mathcal{O}}(L^s)$ is also $\mathcal{O}(L^s)$. The main difference between $\widehat{\mathcal{O}}$ and \mathcal{O} is that for all $i \neq k$ we have

$$I_i = \mathcal{O}(L^0),$$

whereas we had $I_i = \widehat{\mathcal{O}}(L)$.

We want to estimate $H(x(T)) - H(x(0))$ when $T = \alpha L^{2k-3}$, with $\alpha > 0$. Since the dissipation happens only in the variable I_1 , we have

$$H(x(T)) - H(x(0)) = -\gamma \int_0^T I_1^2(t) dt. \quad (7.2)$$

In the new variables and using Lemma 6.6, (7.2) takes the form

$$\begin{aligned} H(x(T)) - H(x(0)) &= -\gamma \int_0^T \widetilde{I}_1^2(t) dt \\ &\quad - \gamma \int_0^T P_1^2(\widetilde{x}(t)) dt \\ &\quad - 2\gamma \int_0^T \widetilde{I}_1(t) P_1(\widetilde{x}(t)) dt. \end{aligned} \quad (7.3)$$

We can say the following about the three contributions above.

- The first integral has a negative sign, but our analysis gives little control over it. Our numerical experiments indicate that the integrand is only significant for a short, initial transient, and that its contribution to the dissipation in the quasi-stationary state is negligible.
- The second integral also has a negative sign, and is of order $\mathcal{O}(TL^{6-4k}) = \mathcal{O}(L^{3-2k})$. This is where the dissipation rate in Theorem 2.4 (ii) comes from. In Sect. 7.1 we will show that under Assumptions 2.1 and 2.2, the integral $\int_0^T P_1^2(\widetilde{x}(t)) dt$ is bounded *below* by a constant times L^{3-2k} , which is crucial.
- The third integral has no sign, and naive dimensional analysis suggests it is $\mathcal{O}(TL^{3-2k}) = \mathcal{O}(L^0)$. However, due to its oscillatory nature, this integral is in fact much smaller. In Sect. 7.2, we use integration by parts (homogenization) to show that it can essentially be reduced to boundary terms of order $\mathcal{O}(L^{2-2k})$, plus higher order corrections, so that it is negligible compared to the first two integrals in (7.3).

Remark 7.2. The choice $T \sim L^{2k-3}$ is made so that the second integral in (7.3), which scales like TL^{6-4k} , dominates the boundary terms of order L^{2-2k} coming from the third integral.

The next two subsections, which deal respectively with the second and third terms in (7.3), will lead to the proof of Theorem 2.4.

7.1. Dissipation is not zero

The main result of this subsection is the following lower bound on the absolute value of the second term in (7.3):

Proposition 7.3. *There exists a constant $c > 0$ such that for all L large enough, and $T = \alpha L^{2k-3}$,*

$$\int_0^T P_1^2(\tilde{x}(t)) dt \geq \frac{\alpha c}{L^{2k-3}}. \quad (7.4)$$

Before we start with the proof of Proposition 7.3, we need some auxiliary material.

Recalling the definition of Q in (4.10), we immediately have

$$\begin{aligned} \partial_{I_i}(Qf) &= -\partial_{I_i} \left(i \sum_{\mu \in \mathcal{N}^{\text{NR}}} \frac{f_\mu(\mathbf{I})}{\mathbf{I} \cdot \mu} e^{i\mu \cdot \varphi} \right) \\ &= i \sum_{\mu \in \mathcal{N}^{\text{NR}}} \left(-\frac{\partial_{I_i} f_\mu(\mathbf{I})}{\mathbf{I} \cdot \mu} + \frac{\mu_i f_\mu(\mathbf{I})}{(\mathbf{I} \cdot \mu)^2} \right) e^{i\mu \cdot \varphi} \\ &= (Q\partial_{I_i} - Q^2\partial_{\varphi_i})f. \end{aligned} \quad (7.5)$$

In the sequel, we often omit the argument $\varphi_{j+1} - \varphi_j$ of U_j and its derivatives.

Lemma 7.4. *We have*

$$\partial_{\varphi_{k-j-1}} f^{(j)} = U''_{k-j-1} Q^2 \partial_{\varphi_{k-j}} f^{(j-1)} + \widehat{\mathcal{O}}^{\text{fin}}(L^{-2j-1}). \quad (7.6)$$

Proof. Recall that by (5.9),

$$f^{(j)} = \left\{ \sum_{s=0}^{j-2} f^{(s,\text{R})} + f^{(j-1)} - \frac{f^{(j-1,\text{NR})}}{2}, \chi^{(j-1)} \right\} + \widehat{\mathcal{O}}^{\text{fin}}(L^{-2j-1}),$$

where the remainder contains finitely many Fourier modes, since $f^{(j)}$ does. The only term in the Poisson bracket involving the site $k-j-1$ is actually in $f^{(0,\text{R})}$ here, since all other terms depend only on the variables $k-j, k-j+1, \dots$. Thus,

$$\partial_{\varphi_{k-j-1}} f^{(j)} = \left\{ \partial_{\varphi_{k-j-1}} f^{(0,\text{R})}, \chi^{(j-1)} \right\} + \widehat{\mathcal{O}}^{\text{fin}}(L^{-2j-1}).$$

Observing that $\partial_{\varphi_{k-j-1}} f^{(0,\text{R})} = U'_{k-j-2}(\varphi_{k-j-1} - \varphi_{k-j-2}) - U'_{k-j-1}(\varphi_{k-j} - \varphi_{k-j-1})$ and that the first contribution has vanishing Poisson bracket with $\chi^{(j-1)}$, we obtain

$$\begin{aligned} \partial_{\varphi_{k-j-1}} f^{(j)} &= \{-U'_{k-j-1}, \chi^{(j-1)}\} + \widehat{\mathcal{O}}^{\text{fin}}(L^{-2j-1}) \\ &= -U''_{k-j-1} \partial_{I_{k-j}} \chi^{(j-1)} + \widehat{\mathcal{O}}^{\text{fin}}(L^{-2j-1}). \end{aligned}$$

But now, $\partial_{I_{k-j}} \chi^{(j-1)} = \partial_{I_{k-j}} Qf^{(j-1)} = (Q\partial_{I_{k-j}} - Q^2\partial_{\varphi_{k-j}})f^{(j-1)}$ by (7.5). But by its definition, $f^{(j-1)}$ depends on φ_{k-j} but not on I_{k-j} (because it is built from some Poisson brackets between $\chi^{(j-2)}$ and some forces, none of which involve I_{k-j}). Thus, we indeed obtain (7.6). \square

Now, using (6.11), the fact that ∂_{φ_1} commutes with Q , and then Lemma 7.4 with $j = k - 2$ leads to

$$\begin{aligned} P_1 &= -\partial_{\varphi_1} \chi^{(k-2)} + \widehat{\mathcal{O}}(L^{5-4k}) \\ &= -Q \partial_{\varphi_1} f^{(k-2)} + \widehat{\mathcal{O}}(L^{5-4k}) \\ &= -Q \left(U_1'' Q^2 \partial_{\varphi_2} f^{(k-3)} + \widehat{\mathcal{O}}^{\text{fin}}(L^{3-2k}) \right) + \widehat{\mathcal{O}}(L^{5-4k}) \\ &= -Q U_1'' Q^2 \partial_{\varphi_2} f^{(k-3)} + \widehat{\mathcal{O}}(L^{2-2k}) . \end{aligned}$$

Using Lemma 7.4 again repeatedly and finally that $\partial_{\varphi_k} f^{(0)} = U'_{k-1}$, we obtain

$$P_1 = -Q U_1'' Q^2 U_2'' Q^2 \cdots U_{k-2}'' Q^2 U'_{k-1} + \widehat{\mathcal{O}}(L^{2-2k}) . \quad (7.7)$$

We now show that P_1 is well approximated by a simple explicit formula. To construct this formula, consider first (7.7) for the special choice $I_k = L$ and $I_i = 0$ for all $i \neq k$. Then Q has the effect of dividing each term of the Fourier series by $i\mu_k L$, *i.e.*, it integrates with respect to φ_k and divides by L . Since U_{k-1} is the only factor that depends on φ_k , the operator Q just integrates U_{k-1} and divides by L . Thus, since the operator Q appears $2k - 3$ times, we expect P_1 to be well approximated by $M_1(\varphi)/L^{2k-3}$ with

$$M_1(\varphi) = -U_1'' U_2'' \cdots U_{k-2}'' G , \quad (7.8)$$

where G is the unique solution of $d^{2k-4}G(\varphi)/d\varphi^{2k-4} = U_{k-1}(\varphi)$ satisfying $\int_0^{2\pi} G(\varphi) d\varphi = 0$. The next lemma shows that the error made by this approximation is only $\mathcal{O}(L^{2-2k})$.

Lemma 7.5. *We have*

$$P_1 = \frac{M_1(\varphi)}{L^{2k-3}} + \mathcal{O}(L^{2-2k}) . \quad (7.9)$$

Proof. Consider the Fourier series of U_i , written

$$U_i(\varphi_{i+1} - \varphi_i) = \sum_{\boldsymbol{\mu} \in \mathcal{N}_i} \widehat{U}_i(\boldsymbol{\mu}) e^{i\boldsymbol{\mu} \cdot \boldsymbol{\varphi}} ,$$

where \mathcal{N}_i is a finite (by Assumption 2.1) set of vectors $\boldsymbol{\mu} \in \mathbf{Z}^n$ satisfying $\mu_j = 0$ for $j \notin \{i, i+1\}$, and $\mu_{i+1} = -\mu_i$. Note that in particular $U_i'(\varphi_{i+1} - \varphi_i) = i \sum_{\boldsymbol{\mu} \in \mathcal{N}_i} \mu_{i+1} \widehat{U}_i(\boldsymbol{\mu}) e^{i\boldsymbol{\mu} \cdot \boldsymbol{\varphi}}$, and similarly for U_i'' with one more factor of $i\mu_{i+1}$. In such notation, observe that (7.7) reads

$$P_1 = - \sum_{\boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(k-1)}} \frac{e^{i\boldsymbol{\mu}^{[1]} \cdot \boldsymbol{\varphi}} (\mu_2^{(1)})^2 \widehat{U}_1(\boldsymbol{\mu}^{(1)})}{\mu_k^{(k-1)} \boldsymbol{\mu}^{[1]} \cdot \mathbf{I}} \prod_{i=2}^{k-1} \frac{(\mu_{i+1}^{(i)})^2 \widehat{U}_i(\boldsymbol{\mu}^{(i)})}{(\boldsymbol{\mu}^{[i]} \cdot \mathbf{I})^2} + \widehat{\mathcal{O}}(L^{2-2k}) , \quad (7.10)$$

where we sum over all $\boldsymbol{\mu}^{(i)} \in \mathcal{N}_i$ for $i = 1, \dots, k-1$, and where we write

$$\boldsymbol{\mu}^{[j]} = \sum_{i=j}^{k-1} \boldsymbol{\mu}^{(i)} .$$

Now, observe that

$$\frac{1}{\boldsymbol{\mu}^{[1]} \cdot \mathbf{I}} \prod_{i=2}^{k-1} \frac{1}{(\boldsymbol{\mu}^{[i]} \cdot \mathbf{I})^2} = \frac{1}{L^{2k-3}} \frac{1}{\mu_k^{[1]}} \prod_{i=2}^{k-1} \frac{1}{(\mu_k^{[i]})^2} + \mathcal{O}(L^{2-2k}), \quad (7.11)$$

as Taylor's theorem and the definition of $\mathcal{B}_{L, \varrho^*}$ show. (Note that here we only have an $\mathcal{O}(L^{2-2k})$ and not an $\hat{\mathcal{O}}(L^{2-2k})$.)

Moreover, observe that for all $i = 1, \dots, k-1$, we have $\mu_k^{[i]} = \mu_k^{(k-1)}$, which is just a manifestation of the fact that U_{k-1} is the only interaction potential involving φ_k . By this observation, (7.11), and the fact that the sum in (7.10) is finite, we obtain

$$P_1 = -\frac{1}{L^{2k-3}} \sum_{\boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(k-1)}} e^{i\boldsymbol{\mu}^{[1]} \cdot \boldsymbol{\varphi}} \frac{\hat{U}_{k-1}(\boldsymbol{\mu}^{(k-1)})}{(\mu_k^{(k-1)})^{2k-4}} \prod_{i=1}^{k-2} (\mu_{i+1}^{(i)})^2 \hat{U}_i(\boldsymbol{\mu}^{(i)}) + \mathcal{O}(L^{2-2k}),$$

from which it is easy to identify (7.9). \square

Looking at (6.14), we expect that when L is large, $\tilde{x}(t)$ will be well approximated by the solution $\bar{x}(t)$ of the following *decoupled ODE*:

$$\begin{aligned} \frac{d}{dt} \bar{I}_i &= -\frac{\partial f^{(0,R)}}{\partial \varphi_i}(\bar{\boldsymbol{\varphi}}) - \gamma \delta_{1,i} \bar{I}_1, \\ \frac{d}{dt} \bar{\varphi}_i &= \bar{I}_i, \end{aligned} \quad (7.12)$$

with $\bar{x}(0) = \tilde{x}(0)$. We call this system *decoupled* since the site k does not interact at all with the other sites (recall that $f^{(0,R)}$ contains all the interaction potentials U_i except those that involve the site k , see (4.6)). In fact, the subsystems $(\bar{\varphi}_\ell, \bar{I}_\ell, \dots, \bar{\varphi}_{k-1}, \bar{I}_{k-1})$, $(\bar{\varphi}_k, \bar{I}_k)$, and $(\bar{\varphi}_{k+1}, \bar{I}_{k+1}, \dots, \bar{\varphi}_n, \bar{I}_n)$ (if $k < n$) are isolated from each others, and in particular \bar{I}_k remains constant.

The next proposition shows that along the trajectories of the decoupled system, the integral $\int_0^1 M_1^2(\bar{\boldsymbol{\varphi}}(t)) dt$ is bounded below. This and the fact that $\tilde{x}(t)$ is close to $\bar{x}(t)$ will lead to the proof of Proposition 7.3.

Lemma 7.6. *The solution to (7.12) is a real-analytic function of t for all times.*

Proof. It follows from the Cauchy-Kovalevskaya theorem that the solution is locally real analytic in time. Since the energy of the decoupled system is non-increasing, the solution cannot blow up, and the Picard existence and uniqueness theorem guarantees that the solution is real-analytic for all times. \square

Proposition 7.7. *Under the decoupled dynamics (7.12), there is a constant $\bar{c}_* > 0$ such that if L is large enough, then for any initial condition $\bar{x}(0) = (\bar{\mathbf{I}}(0), \bar{\boldsymbol{\varphi}}(0)) \in \mathcal{B}_{L, \varrho^*}$, we have*

$$\int_0^1 M_1^2(\bar{\boldsymbol{\varphi}}(t)) dt \geq \bar{c}_*. \quad (7.13)$$

Proof. Let

$$K(t) = U_1''(\bar{\varphi}_2(t) - \bar{\varphi}_1(t)) \cdot U_2''(\bar{\varphi}_3(t) - \bar{\varphi}_2(t)) \cdots U_{k-2}''(\bar{\varphi}_{k-1}(t) - \bar{\varphi}_{k-2}(t)) .$$

First, observe that there is a constant C , independent of L , such that for all $\bar{x}(0) \in \bar{\mathcal{B}}_{L,\varrho^*}$ (the closure of $\mathcal{B}_{L,\varrho^*}$) and all $t \in [0, 1]$ we have $|K(t)| \leq C$ and $|\dot{K}(t)| \leq C$.

Next, we have

$$M_1^2(\bar{\varphi}(t)) = K^2(t)G^2(\bar{\varphi}_k(t)) = K^2(t)(\langle G^2 \rangle + \psi'(\bar{\varphi}_k(t))) ,$$

where

$$\langle G^2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} G^2(\varphi) d\varphi > 0 ,$$

and ψ is a primitive of $G^2 - \langle G^2 \rangle$ on \mathbf{T} (the fact that $\langle G^2 \rangle$ is strictly positive follows from Assumption 2.2, which guarantees that U_{k-1} , and hence also G , are non-constant).

Thus, recalling that $\frac{d}{dt}\bar{\varphi}_k(t) = \bar{I}_k(t) = \bar{I}_k(0)$, we obtain after integrating by parts that

$$\int_0^1 K^2(t)\psi'(\bar{\varphi}_k(t)) dt = \left[\frac{\psi(\bar{\varphi}_k(t))}{\bar{I}_k(0)} K^2(t) \right]_0^1 - 2 \int_0^1 \frac{\psi(\bar{\varphi}_k(t))}{\bar{I}_k(0)} K(t) \dot{K}(t) dt .$$

Since both terms above are $\mathcal{O}(L^{-1})$, we have

$$\int_0^1 M_1^2(\bar{\varphi}(t)) dt = \langle G^2 \rangle \int_0^1 K^2(t) dt + \mathcal{O}(L^{-1}) .$$

Observe now that $\inf_{\bar{x}(0) \in \mathcal{B}_{L,\varrho^*}} \int_0^1 K^2(t) dt$ is independent of L , since $K(t)$ does not depend on I_k . Thus, since the $\mathcal{O}(L^{-1})$ above can be made arbitrarily small, the proposition is proved if we can show that

$$\inf_{\bar{x}(0) \in \mathcal{B}_{0,\varrho^*}} \int_0^1 K^2(t) dt > 0 .$$

By compactness, it suffices to prove that for all $\bar{x}(0) \in \bar{\mathcal{B}}_{0,\varrho^*}$,

$$\int_0^1 K^2(t) dt > 0 . \tag{7.14}$$

Assume by contradiction that $\bar{x}(0) \in \bar{\mathcal{B}}_{0,\varrho^*}$ and that $K(t) \equiv 0$ on the interval $[0, 1]$. By analyticity (Lemma 7.6), it is easy to realize that this implies that for some $\ell < k$, $U_\ell''(\bar{\varphi}_\ell(t) - \bar{\varphi}_{\ell-1}(t)) \equiv 0$ for all $t > 0$ (not only in $[0, 1]$). We now show that this leads to a contradiction.

Since U_ℓ'' has no flat part by Assumption 2.1, this implies that actually $\bar{\varphi}_\ell(t) - \bar{\varphi}_{\ell-1}(t)$ is constant. By Assumption 2.2, we have $U_\ell'(\bar{\varphi}_\ell(t) - \bar{\varphi}_{\ell-1}(t)) \equiv \varkappa \neq 0$. But then, the subsystem $(\bar{\varphi}_\ell, \bar{I}_\ell, \dots, \bar{\varphi}_{k-1}, \bar{I}_{k-1})$ receives a constant, non-zero force \varkappa . Thus, the total momentum of this subsystem will eventually be arbitrarily large, and so will its energy. This is a contradiction, since the energy of the system $(\bar{\varphi}_1, \bar{I}_1, \dots, \bar{\varphi}_{k-1}, \bar{I}_{k-1})$ is non-increasing. The proof is complete. \square

The next proposition shows that these estimates extend to the dynamics of \tilde{x} , and also to any subinterval of length 1 in $[0, T]$.

Proposition 7.8. *There is a constant $\tilde{c}_* > 0$ such that for all large enough L , and all $0 \leq t_0 \leq T - 1$,*

$$\int_{t_0}^{t_0+1} M_1^2(\tilde{\varphi}(t)) dt \geq \tilde{c}_* . \quad (7.15)$$

Proof. We compare the trajectory $\tilde{x}(t_0 + t)$ and the solution $\bar{x}(t)$ of (7.12) with initial condition $\bar{x}(0) = \tilde{x}(t_0)$, which belongs to $\mathcal{B}_{L, \varrho^*}$. By comparing (6.14) and (7.12), we see that

$$\sup_{t \in [0, 1]} |\tilde{x}(t_0 + t) - \bar{x}(t)| = \mathcal{O}(L^{-2}) .$$

Using Proposition 7.7, we then obtain the desired result with $\tilde{c}_* = \bar{c}_*/2$ and L sufficiently large, since M_1 is uniformly continuous. \square

Proof of Proposition 7.3. By Lemma 7.5,

$$\begin{aligned} \int_0^T P_1^2(\tilde{x}(t)) dt &\geq \frac{1}{L^{4k-6}} \int_0^T M_1^2(\tilde{\varphi}(t)) dt + \mathcal{O}(T/L^{4k-5}) \\ &= \frac{1}{L^{4k-6}} \int_0^T M_1^2(\tilde{\varphi}(t)) dt + \mathcal{O}(L^{2-2k}) . \end{aligned} \quad (7.16)$$

Next, we consider the decomposition

$$\int_0^T M_1^2(\tilde{\varphi}(t)) dt \geq \sum_{n=0}^{\lfloor T-1 \rfloor} \int_n^{n+1} M_1^2(\tilde{\varphi}(t)) dt .$$

By Proposition 7.8, we find that for all large enough L ,

$$\int_0^T M_1^2(\tilde{\varphi}(t)) dt \geq \tilde{c}_* \lfloor T \rfloor .$$

Thus, by (7.16), we indeed get (7.4) if $c = \tilde{c}_*/2$ and L is large enough. This completes the proof. \square

7.2. Neglecting the mixed term

In this subsection we complete the proof of Theorem 2.4. Note that (7.3) and Proposition 7.3 imply that

$$H(x(T)) - H(x(0)) \leq -\gamma \int_0^T \tilde{I}_1^2(t) dt - \frac{\gamma \alpha c}{L^{2k-3}} - 2\gamma \int_0^T \tilde{I}_1(t) P_1(\tilde{x}(t)) dt . \quad (7.17)$$

It remains to prove that the last term in the right-hand side is smaller (in magnitude) than the first two, from which Theorem 2.4 will follow.

First, by (6.11), we can write

$$P_1 = P_{10} + P_{11} ,$$

where

$$\begin{aligned} P_{10} &= -\partial_{\varphi_1} \chi^{(k-2)} = \widehat{\mathcal{O}}_{\text{NR}}^{\text{fin}}(L^{3-2k}) , \\ P_{11} &= \widehat{\mathcal{O}}(L^{5-4k}) . \end{aligned}$$

By Young's inequality, for any $K > 0$,

$$|2\widetilde{I}_1 P_{11}(\widetilde{x})| \leq \frac{\widetilde{I}_1^2}{K} + K P_{11}^2(\widetilde{x}) .$$

Thus, we find

$$-2\gamma \int_0^T \widetilde{I}_1(t) P_{11}(\widetilde{x}(t)) dt \leq \frac{1}{K} \int_0^T \widetilde{I}_1^2(t) dt + \mathcal{O}(L^{7-6k}) . \quad (7.18)$$

If we choose first K large enough, and then L large enough, the above is indeed smaller than the two “good” terms in (7.17).

It remains therefore only to deal with

$$\int_0^T \widetilde{I}_1(t) P_{10}(\widetilde{x}(t)) dt . \quad (7.19)$$

The main result of this subsection is the following proposition:

Proposition 7.9. *We have*

$$\int_0^T \widetilde{I}_1(t) P_{10}(\widetilde{x}(t)) dt = \mathcal{O}(L^{2-2k}) . \quad (7.20)$$

This proposition, together with (7.18), will guarantee that the third term in (7.17) is small in comparison with the first two. We will prove Proposition 7.9 by successive integrations by parts.

The integrand in (7.19) has the form $\widetilde{I}_1 \widehat{\mathcal{O}}_{\text{NR}}^{\text{fin}}(L^{3-2k})$. Our main induction step deals with slightly more general integrands, where the power of \widetilde{I}_1 can be different from one. We say that an admissible function f is of class $R(p, m)$ with $p, m \geq 0$ if

$$f(x) = I_1^m f^*(x) ,$$

where $f^* = \widehat{\mathcal{O}}_{\text{NR}}^{\text{fin}}(L^{-p})$. Note that we have $f = \mathcal{O}(L^{-p})$, but only $f = \widehat{\mathcal{O}}(L^{m-p})$.

Lemma 7.10. *Let f be of class $R(p, m)$ with $m \geq 0$ and $p \geq 2k - 3$. Then, there exist finitely many functions g_1, \dots, g_N , each of class $R(m_\ell, p_\ell)$ with $m_\ell \geq 0$ and $p_\ell \geq p + 1$, such that*

$$\int_0^T f(\widetilde{x}(t)) dt = \sum_{\ell=1}^N \int_0^T g_\ell(\widetilde{x}(t)) dt + \mathcal{O}(L^{2-2k}) .$$

Proof. Since f is of class $R(p, m)$, we have $f = I_1^m f^*$, where $f^* = \widehat{\mathcal{O}}_{\text{NR}}^{\text{fin}}(L^{-p})$. We can thus define

$$F(x) = Qf(x) = I_1^m Qf^*(x) ,$$

which is of class $R(p+1, m)$, and hence $\mathcal{O}_{\text{NR}}^{\text{fin}}(L^{-p-1})$. The key observation is that

$$\sum_{i=1}^n I_i \partial_{\varphi_i} F = I_1^m \sum_{i=1}^n I_i \partial_{\varphi_i} Qf^* = I_1^m f^* = f . \quad (7.21)$$

(The above is obtained like (4.11), recalling that f^* is non-resonant.)

Moreover, by Lemma 6.8,

$$\begin{aligned} \frac{d}{dt} F(\tilde{x}(t)) &= \sum_{i=1}^n \frac{\partial F}{\partial \varphi_i}(\tilde{x}(t)) \left(\tilde{I}_i(t) + \mathcal{O}_{\text{R}}^{\text{fin}}(L^{-3}) + \mathcal{O}(L^{2-2k}) \right) \\ &\quad + \sum_{i=1}^n \frac{\partial F}{\partial I_i}(\tilde{x}(t)) \left(\mathcal{O}_{\text{R}}^{\text{fin}}(L^0) + \mathcal{O}(L^{3-2k}) - \gamma \delta_{1,i} \tilde{I}_1(t) \right) \\ &= \sum_{i=1}^n \tilde{I}_i(t) \frac{\partial F}{\partial \varphi_i}(\tilde{x}(t)) + \mathcal{O}_{\text{NR}}^{\text{fin}}(L^{-p-1}) + \tilde{I}_1(t) \mathcal{O}_{\text{NR}}^{\text{fin}}(L^{-p-1}) + \mathcal{O}(L^{5-4k}) , \end{aligned}$$

where we have used that $\partial_{\varphi_i} F$ and $\partial_{I_i} F$ are $\mathcal{O}_{\text{NR}}^{\text{fin}}(L^{-p-1})$, that $p \geq 2k-3$, and that the product of a non-resonant term and a resonant one is non-resonant. Using now (7.21) and decomposing the two $\mathcal{O}_{\text{NR}}^{\text{fin}}(L^{-p-1})$ above, we find

$$\frac{d}{dt} F(\tilde{x}(t)) = f(\tilde{x}(t)) - \sum_{\ell=1}^N g_{\ell}(\tilde{x}(t)) + \mathcal{O}(L^{5-4k}) ,$$

where g_1, \dots, g_N are as in the statement of the lemma. But then, integrating both sides gives

$$\int_0^T f(\tilde{x}(t)) dt = F(\tilde{x}(t))|_0^T + \sum_{\ell=1}^N \int_0^T g_{\ell}(\tilde{x}(t)) dt + \mathcal{O}(L^{2-2k}) ,$$

from which the result follows since, $F = \mathcal{O}(L^{-p-1}) = \mathcal{O}(L^{2-2k})$. \square

Proof of Proposition 7.9. The integrand in (7.20) is of class $R(2k-3, 1)$. The result then immediately follows from applying Lemma 7.10 recursively. Indeed, at each step, we have finitely many functions g_{ℓ} . Moreover, at each step the degree p_{ℓ} of these functions increases at least by one. Thus, after finitely many steps all the g_{ℓ} have order $p_{\ell} \geq 5-4k$, and hence $\int_0^T g_{\ell}(\tilde{x}(t)) dt = \mathcal{O}(TL^{5-4k}) = \mathcal{O}(L^{2-2k})$, which completes the proof. \square

We can finally prove the main theorem.

Proof of Theorem 2.4. Part (i) was proved in Proposition 6.9, and Part (ii) follows immediately from the estimates above. Indeed, putting together (7.17), (7.18) (with K large enough) and (7.20) implies that

$$H(x(T)) - H(x(0)) \leq -\frac{\gamma \alpha c}{L^{2k-3}} + \mathcal{O}(L^{2-2k}) ,$$

which, by setting $C_1 = c/2$ and using the definition of \mathcal{O} , completes the proof of Part (ii). \square

8. Asymptotic equations

We have found, through computer simulations and the arguments in Sect. 3, that under dissipation, the system quickly approaches a quasi-stationary state where I_k oscillates with very small amplitude around some fixed value $\langle I_k \rangle \approx L$, and where all other I_j oscillate with very small amplitude around zero. While we have not been able to prove that this quasi-stationary state exists, it is still instructive to derive asymptotic equations which seem to describe it. We assume here that the potentials are $U_i(\varphi_{i+1} - \varphi_i) = -\cos(\varphi_{i+1} - \varphi_i)$.

Since the results of the previous sections show that the value of the action variable $I_k(t)$ changes very slowly, we have

$$\dot{\varphi}_k = I_k \approx L ,$$

to a high degree of accuracy. Thus, we can, for long times, approximate

$$\varphi_k(t) \approx Lt , \quad I_k(t) \approx L .$$

(For simplicity, we have assumed that $\varphi_k(0) = 0$.)

Now consider the evolution of I_i for $i < k$, which is given by

$$\begin{aligned} \dot{I}_{k-1} &= \sin(\varphi_{k-2} - \varphi_{k-1}) - \sin(\varphi_{k-1} - \varphi_k) , \\ \dot{I}_{k-2} &= \sin(\varphi_{k-3} - \varphi_{k-2}) - \sin(\varphi_{k-2} - \varphi_{k-1}) , \\ &\dots \\ \dot{I}_1 &= \sin(\varphi_2 - \varphi_1) - \gamma I_1 . \end{aligned}$$

Our numerical experiments show that the φ_i , $i < k$, oscillate around some (common) value, which is a minimum of the interactions potentials. Since this minimum is degenerate (as the system is invariant under global rotation), we assume without loss of generality that the φ_i , $i < k$ oscillate around zero. Our numerical experiments show that, in terms of amplitudes of the oscillations,

$$1 \gg \varphi_{k-1} \gg \varphi_{k-2} \gg \dots \gg \varphi_1 .$$

Thus, the equation of motion for I_{k-1} can be approximated by

$$\dot{I}_{k-1} = \sin(\varphi_{k-2} - \varphi_{k-1}) - \sin(\varphi_{k-1} - \varphi_k) \approx \sin(\varphi_k) \approx \sin(Lt) ,$$

so that

$$I_{k-1}(t) \approx -\frac{1}{L} \cos(Lt) .$$

Since

$$\dot{\varphi}_{k-1} = I_{k-1} ,$$

we immediately find

$$\varphi_{k-1}(t) \approx -\frac{1}{L^2} \sin(Lt) .$$

Continuing in this fashion, we arrive at an approximation to the actions and angles in the quasi-stationary state of the form:

$$I_k \approx L ,$$

$$\begin{aligned}
I_{k-1} &\approx -\cos(Lt)/L, \\
\varphi_{k-1} &\approx -\sin(Lt)/L^2, \\
I_{k-2} &\approx \cos(Lt)/L^3, \\
\varphi_{k-2} &\approx \sin(Lt)/L^4, \\
I_{k-3} &\approx -\cos(Lt)/L^5, \\
\varphi_{k-3} &\approx -\sin(Lt)/L^6, \\
&\dots
\end{aligned}$$

This scaling predicted by this simple argument corresponds to (3.3) and is corroborated by the results of the numerical calculations, as shown in Fig. 1 and Fig. 2. A similar argument applies to the sites $i > k$.

Note also that, in fact *any* periodic function with non-degenerate quadratic extrema can replace the cosine above.

We conjecture that a whole family of such quasi-stationary states exists, and that after a “fast” initial transient the subsequent dissipative evolution occurs by a slow motion along this family of quasi-stationary solutions. We expect the family of quasi-stationary states to be parameterized by $\langle I_k \rangle$ and the equilibrium position around which the $\varphi_i, i \neq k$ oscillate.

9. Decay rate for degenerate potentials

In Theorem 2.4 we proved a bound on the rate of dissipation of energy for certain trajectories that indicates that the energy loss per unit time should scale like $-L^{6-4k}$ when the k 'th rotator ($k \geq 2$) initially has very large energy and all other rotators have initial energy of order 1. Our numerics indicate that for a system with cosine nearest neighbor potentials, this decay rate is sharp. However, we expect that if the potentials U_i violate Assumption 2.2, the dissipation rate can be much slower. Note, in particular, that this happens when the potentials have degenerate extrema, *i.e.*, $U_i''(\varphi) = 0$ and $U_i'(\varphi) = 0$ for some φ and i .

To illustrate the reasons for our expectations (in a non-rigorous way), consider a chain of 3 rotators, in which the first and the second are connected by a degenerate potential, $U(\varphi) = (\cos(\varphi) - 1)^2/2$, while the second and third rotators are still coupled by a cosine potential. The Hamiltonian is then

$$H = \sum_{i=1}^3 \frac{I_i^2}{2} - \cos(\varphi_3 - \varphi_2) + \frac{(\cos(\varphi_2 - \varphi_1) - 1)^2}{2}.$$

The point here is that the minimum of $U(\varphi) = (\cos(\varphi) - 1)^2/2$ is attained at $\varphi = 0$, and that the first 3 derivatives at 0 vanish, leaving us with the expansion $U(\varphi) = \frac{1}{8}\varphi^4 + \mathcal{O}(\varphi^6)$.

We now proceed to find approximate solutions of the equations of motion following the method sketched in Sect. 8, making the same technical assumptions. We find

$$I_3(t) \approx L, \quad \text{and} \quad \varphi_3(t) \approx Lt.$$

Likewise, we have

$$I_2(t) \approx -\frac{1}{L} \cos(Lt) , \text{ and } \varphi_2(t) \approx -\frac{1}{L^2} \sin(Lt) .$$

Now, consider the equation for $I_1(t)$. Unlike above, we have

$$\begin{aligned} \dot{I}_1 &= [1 - \cos(\varphi_2 - \varphi_1)] \sin(\varphi_2 - \varphi_1) - \gamma I_1 \\ &\approx [1 - \cos(\varphi_2)] \sin(\varphi_2) \approx \frac{\varphi_2^3}{2} , \end{aligned}$$

where the second line appealed to our numerical observation that φ_2 is much larger than φ_1 and I_1 . If we then insert our approximation $\varphi_2(t) \approx -\frac{1}{L^2} \sin(Lt)$, we find

$$\dot{I}_1(t) \approx -\frac{\sin^3(Lt)}{2L^6} ,$$

from which we obtain

$$I_1 \approx \frac{1}{2L^7} \left(\cos(Lt) - \frac{1}{3} \cos^3(Lt) \right) . \quad (9.1)$$

Thus, we find that in contrast to what we found under Assumption 2.2, the action variables now have the following ratios:

$$\frac{|I_2|}{|I_3|} \sim \frac{1}{L^2} , \quad \frac{|I_1|}{|I_2|} \sim \frac{1}{L^6} . \quad (9.2)$$

By (9.1), we expect the dissipation rate to scale like $I_1^2 \sim L^{-14}$, which is much smaller than the $L^{6-4k} = L^{-6}$ that our main theorem gives under Assumption 2.2.

We checked the statements above numerically for $L = 10, 20$ with $\gamma = 0.1$. In Fig. 4 we see the ratios in (9.2) to a very good approximation (observing that the bracket in (9.1) has an amplitude of $2/3$). We have also checked that, in this case, the dissipation rate indeed scales like L^{-14} if we start from an initial condition that is in the quasi-stationary state.

Note finally that one can make the situation much worse: by choosing a potential $(1 - \cos(\varphi_2 - \varphi_1))^r$, we would obtain $I_1 \sim L^{1-4r}$.

A. Bounds on analytic functions

The most convenient way to bound the actions of e^χ on a function f is provided by the analytic methods used by [32]. We adapt them here to our needs, where, instead of the usual small parameter, we have here large momentum I_k .

We work in the domains $\mathcal{D}_{L,r,\sigma}$ defined in Definition 4.1, and use the norm (defined in (4.1)):

$$\|f\|_{L,r,\sigma} = \sup_{\mathcal{D}_{L,r,\sigma}} |f(\mathbf{I}, \boldsymbol{\varphi})| .$$

Remark A.1. We will tacitly assume throughout that $\sigma \leq 1$.

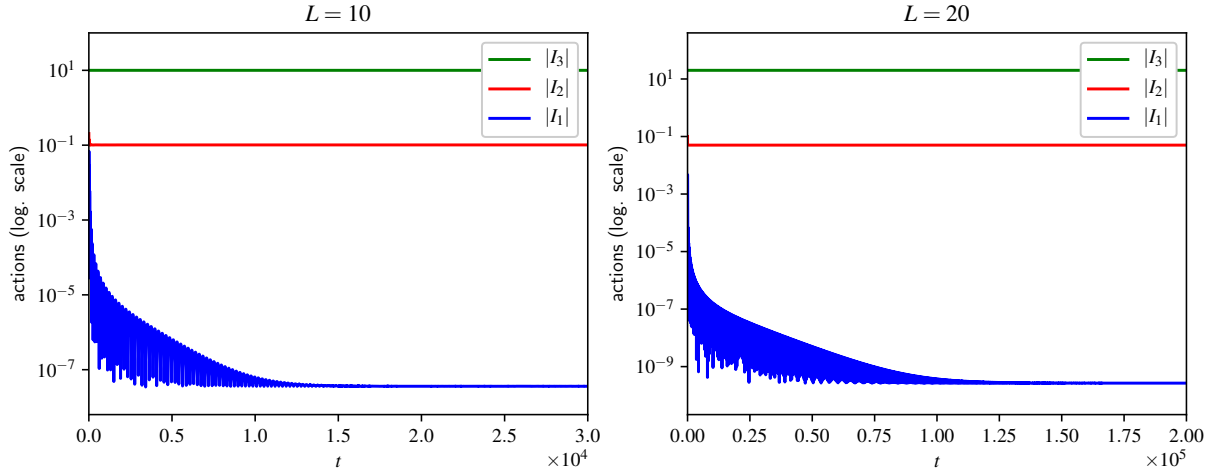


Figure 4. Maximum of $|I_i|$ over intervals of length $2\pi/L$ when the potential U_1 is degenerate, for $n = k = 3$ and $\gamma = 0.1$, with $L = 10$ (left) and $L = 20$ (right). Note that now the ratios in the quasi-stationary state are $|I_1/I_2| \sim L^{-6}$ and $|I_2/I_3| \sim L^{-2}$.

We also assume that the functions f, f_1, f_2, \dots have the form (4.2). The next lemma bounds restrictions of Fourier series.

Lemma A.2. *If f is analytic on $\mathcal{D}_{L,r,\sigma}$, then for any subset $\tilde{\mathcal{N}} \subset \mathcal{N}$, the function $\tilde{f} = \sum_{\mu \in \tilde{\mathcal{N}}} f_\mu(\mathbf{I}) e^{i\mu \cdot \varphi}$ is analytic on $\mathcal{D}_{L,r,\tilde{\sigma}}$ for any $0 < \tilde{\sigma} < \sigma$ and*

$$\|\tilde{f}\|_{L,r,\tilde{\sigma}} \leq \left(\frac{4}{\sigma - \tilde{\sigma}} \right)^n \|f\|_{L,r,\sigma}.$$

Proof. Since f is analytic (in φ), we have $|f_\mu(\mathbf{I})| \leq \|f\|_{L,r,\sigma} e^{-\sigma|\mu|}$, and therefore

$$\|\tilde{f}\|_{L,r,\tilde{\sigma}} \leq \sum_{\mu \in \tilde{\mathcal{N}}} \|f\|_{L,r,\sigma} e^{-(\sigma - \sigma')|\mu|}.$$

Note now that with $\delta = \sigma - \sigma' > 0$, and since by Remark A.1, $\delta \leq 1$, one has

$$\sum_{\mu \in \tilde{\mathcal{N}}} e^{-\delta|\mu|} \leq \prod_{i=1}^n \left(\sum_{\mu_i=-\infty}^{\infty} e^{-\delta|\mu_i|} \right) \leq \left(\frac{2}{1 - e^{-\delta}} \right)^n \leq (4/\delta)^n. \quad (\text{A.1})$$

□

Proposition A.3. *Suppose f is analytic on $\mathcal{D}_{L,r,\sigma}$ and that the decomposition (4.2) contains finitely many Fourier modes. If $r < 1/(20n|\mathcal{N}|)$ (recall Definition 4.4), the function Qf defined in (4.10) is analytic on $\mathcal{D}_{L,r,\sigma'}$ for any $0 < \sigma' < \sigma$, and*

$$\|Qf\|_{L,r,\sigma'} \leq 2 \left(\frac{4}{\sigma - \sigma'} \right)^n \frac{\|f\|_{L,r,\sigma}}{L}.$$

Proof. By Lemma 4.7, each summand in the definition of Qf is analytic on $\mathcal{D}_{L,r,\sigma'}$. Since the sum is finite, Qf is analytic on the same domain. Since $|f_\mu(\mathbf{I})| \leq \|f\|_{L,r,\sigma} e^{-\sigma|\mu|}$, and by

Lemma 4.7, we find

$$\|Qf\|_{L,r,\sigma'} \leq \sum_{\mu \in \mathcal{N}^{\text{NR}}} \frac{\|f\|_{L,r,\sigma} e^{-(\sigma-\sigma')|\mu|}}{L/2}.$$

The assertion follows using again (A.1). \square

Lemma A.4. *Suppose f_1 and f_2 are analytic on $\mathcal{D}_{L,r_1,\sigma_1}$ and $\mathcal{D}_{L,r_2,\sigma_2}$ respectively. Then $\{f_1, f_2\}$ is analytic on $\mathcal{D}_{\min(r_1,r_2),\min(\sigma_1,\sigma_2)}$. Moreover, if $r' < \min(r_1, r_2)$ and $\sigma' < \min(\sigma_1, \sigma_2)$, then one has the bound*

$$\|\{f_1, f_2\}\|_{L,r',\sigma'} \leq \left(\frac{n}{L(r_1 - r')(\sigma_2 - \sigma')} + \frac{n}{L(r_2 - r')(\sigma_1 - \sigma')} \right) \times \|f_1\|_{L,r_1,\sigma_1} \|f_2\|_{L,r_2,\sigma_2}.$$

Proof. By Cauchy's theorem, for $s = 1, 2$, and $i = 1, \dots, n$,

$$\begin{aligned} \left\| \frac{\partial f_s}{\partial I_i} \right\|_{L,r',\sigma_s} &\leq \frac{1}{L(r_s - r')} \|f_s\|_{L,r_s,\sigma_s}, \\ \left\| \frac{\partial f_s}{\partial \varphi_i} \right\|_{L,r_s,\sigma'} &\leq \frac{1}{\sigma_s - \sigma'} \|f_s\|_{L,r_s,\sigma_s}. \end{aligned} \tag{A.2}$$

We have

$$\{f_1, f_2\} = \sum_{i=1}^n \frac{\partial f_1}{\partial \varphi_i} \frac{\partial f_2}{\partial I_i} - \frac{\partial f_1}{\partial I_i} \frac{\partial f_2}{\partial \varphi_i}.$$

Since f_1 and f_2 are analytic on $\mathcal{D}_{L,r,\sigma}$, so are their derivatives and $\{f_1, f_2\}$ is a finite sum of analytic functions and hence analytic on $\mathcal{D}_{\min(r_1,r_2),\min(\sigma_1,\sigma_2)}$. The bound on the norm comes from applying (A.2) to the $2n$ terms in the sum. \square

We have the immediate

Corollary A.5. *Let f_1 and f_2 be analytic on $\mathcal{D}_{L,r,\sigma}$. Then, for all $0 < \sigma' < \sigma$ and $0 < r' < r$, we have*

$$\|\{f_1, f_2\}\|_{L,r',\sigma'} \leq \frac{2n}{L(r - r')(\sigma - \sigma')} \|f_1\|_{L,r,\sigma} \|f_2\|_{L,r,\sigma}.$$

To sum the Lie series, we follow Pöschel [33]. We begin by bounding ad^ℓ

Lemma A.6. *Assume g and f are analytic on $\mathcal{D}_{L,r,\sigma}$. Then for all $\ell \geq 1$,*

$$\|\text{ad}_g^\ell f\|_{L,r',\sigma'} \leq \left(\frac{4n\ell}{L(r - r')(\sigma - \sigma')} \right)^\ell \|g\|_{L,r,\sigma}^\ell \|f\|_{L,r,\sigma}. \tag{A.3}$$

Proof. We fix $\ell \geq 1$ and estimate $\text{ad}_g^\ell f$ using a sequence of nested domains. For $s = 0, 1, 2, \dots, \ell$, we write

$$z_s \equiv (r_s, \sigma_s) \equiv \left(r' + \frac{s(r - r')}{2\ell}, \sigma' + \frac{s(\sigma - \sigma')}{2\ell} \right).$$

Applying Lemma A.4 we find for $s = 0, \dots, \ell - 1$,

$$\begin{aligned} \|\text{ad}_g^{\ell-s} f\|_{L, z_s} &\leq \left(\frac{n}{L(r-r_s)(\sigma_{s+1}-\sigma_s)} + \frac{n}{L(r_{s+1}-r_s)(\sigma-\sigma_s)} \right) \\ &\quad \times \|g\|_{L, r, \sigma} \|\text{ad}_g^{s-\ell-1} f\|_{L, z_{s+1}} \\ &\leq \frac{4n\ell}{L(r-r')(\sigma-\sigma')} \|g\|_{L, r, \sigma} \|\text{ad}_g^{\ell-s-1} f\|_{L, z_{\ell+1}}, \end{aligned}$$

where we have used that $(r-r_s) \geq (r-r')/2$, and $(\sigma-\sigma_s) \geq (\sigma-\sigma')/2$, while $r_{s+1}-r_s = (r-r')/\ell$ and $\sigma_{s+1}-\sigma_s = (\sigma-\sigma')/\ell$. Iterating this we get

$$\|\text{ad}_g^\ell f\|_{L, r', \sigma'} = \|\text{ad}_g^\ell f\|_{L, z_0} \leq \left(\frac{4n\ell}{L(r-r')(\sigma-\sigma')} \right)^\ell \|g\|_{L, r, \sigma}^\ell \|f\|_{L, r, \sigma},$$

where we have also used that $\|f\|_{L, z_\ell} \leq \|f\|_{L, r, \sigma}$. \square

Proposition A.7. Assume that $\|g\|_{L, r, \sigma} < \infty$ and $\|f\|_{L, r, \sigma} < \infty$ for some $r, \sigma > 0$ and all large enough L . Moreover, let $(a_\ell)_{\ell \geq 0}$ be a bounded sequence. Then, for all $0 < \sigma' < \sigma$, $0 < r' < r$, and sufficiently large L we have for $0 \leq \ell_0 \leq \ell_1 \leq \infty$,

$$\left\| \sum_{\ell=\ell_0}^{\ell_1} \frac{a_\ell}{\ell!} \text{ad}_g^\ell f \right\|_{L, r', \sigma'} \leq 2 \left(\frac{4ne}{(\sigma-\sigma')(r-r')} \frac{\|g\|_{L, r, \sigma}}{L} \right)^{\ell_0} \|f\|_{L, r, \sigma} \cdot \sup_{\ell \geq 0} |\alpha_\ell|.$$

Proof. Without loss of generality we assume that $\sup_{\ell \geq 0} |\alpha_\ell| = 1$. Using (A.3) and the bound $\ell^\ell / \ell! \leq e^\ell$, we get

$$\begin{aligned} \left\| \sum_{\ell=\ell_0}^{\ell_1} \frac{a_\ell}{\ell!} \text{ad}_g^\ell f \right\|_{L, r', \sigma'} &\leq \sum_{\ell=\ell_0}^{\infty} \frac{(4n\ell)^\ell}{\ell! (L(\sigma-\sigma')(r-r'))^\ell} \|g\|_{L, r, \sigma}^\ell \|f\|_{L, r, \sigma} \\ &\leq \sum_{\ell=\ell_0}^{\infty} \left(\frac{4ne}{(\sigma-\sigma')(r-r')} \frac{\|g\|_{L, r, \sigma}}{L} \right)^\ell \|f\|_{L, r, \sigma} \\ &= \left(\frac{4ne}{(\sigma-\sigma')(r-r')} \frac{\|g\|_{L, r, \sigma}}{L} \right)^{\ell_0} \|f\|_{L, r, \sigma} \times \\ &\quad \sum_{\ell=0}^{\infty} \left(\frac{4ne}{(\sigma-\sigma')(r-r')} \frac{\|g\|_{L, r, \sigma}}{L} \right)^\ell, \end{aligned}$$

which gives the desired result provided L is sufficiently large so that the series in the last line is bounded by 2. Note that the bound is decreasing when L increases.

In case $\ell_1 = \infty$, the result is indeed an analytic function, as the series converges uniformly on the open complex domain $\mathcal{D}_{L, r', \sigma'}$. \square

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